Content Inclusion Logics:

A Philosophical and Logical Investigation

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Contents

Introduction

This dissertation focuses on a family of formal systems known as *variable inclusion logics*, or simply *containment logics*. Containment logics have been developed and explored for a variety of reasons: some were designed to offer a systematic way of dealing with meaningless or nonsensical statements; some have been viewed through epistemic lenses as systems that implement different policies for pooling experts' opinions; some received a computational interpretation as logics that model different kinds of errors in computer programs. Here the focus is on one specific interpretation for such logics, that is, as logics of content inclusion. This motivation can be employed to place these systems within the broader class of relevance logics, insofar as they align with them both in some of the formal tools employed and in their underlying motivation.

The central objective of this dissertation is to provide a systematic study of logics of content inclusion at two levels of analysis. On the one hand, it seeks to survey the existing literature on content inclusion, assessing the respective advantages and limitations of different philosophical accounts of relevance, content, and their application to the notion of implication. On the other hand, it aims to extend the formal semantic study of these systems by exploring the potential of both established and novel frameworks. The overarching goal is to conduct the philosophical and logical investigations in tandem, allowing each to inform and guide the other.

The structure of the dissertation is as follows. Chapter 1 introduces the conceptual apparatus, examining how content inclusion can be understood and how it connects with the relevance tradition. Chapter 2 presents the technical tools, with a particular focus on infectious logics, and clarifies in what sense implication may satisfy the previously outlined notion of content inclusion. Chapter 3 discusses several proposals that can be interpreted as capturing content inclusion in a more refined and, arguably, more adequate way. This aim further motivates the development of novel substructural containment logics. Finally, chapter 4 draws on the full potential of so-called two-address semantics to characterize structural logics of content inclusion and explores the employment of non-determinism.

Part of the content of chapters 3 and 4 draws on joint works with Agustina Borzi and Damian Szmuc. Borzi and Zirattu, 2026b and Borzi and Zirattu, 2026a have been accepted for publication, respectively, in the *Australasian Journal of Logic* and in the *Journal of Logic, Language and Information*, while the latter, Szmuc and Zirattu, 2025, has been published in *Erkenntnis*.

Chapter 1

The Notion of Relevance in Logic

Content, Topics, Relevance

The notion of *relevance* has been widely debated among logicians in the past century, although its roots stretch back to the dawn of logical inquiry. This notion has been invoked most famously as a way to avoid certain validities of the material and strict implication that were deemed paradoxical. The paradoxicality of such validities was attributed by many authors to the lack of relevance, roughly intended as a meaningful connection between antecedent and consequent of conditionals. Many of the attempts to reintroduce such connection gave rise to the so-called relevance logics.

Although the details of relevance logics will not be discussed, it is useful to outline the key intuitions behind these systems as formal frameworks incorporating topic-theoretic concerns into logical validity. The notion of content inclusion is then introduced as a way to strengthen relevance and the philosophical foundations of relevance are assessed with an eye on the notions of content and subject matter.

1.1 Relevance

Relevance can be understood as a relation between the antecedent and consequent of a conditional, or, at the level of inferences, between the premise and conclusion of an argument[1]. This notion was first studied in connection with the former, although a valid conditional satisfying this property was thought of as internalizing some notion of entailment (Anderson and Belnap, 1975; see also Avron, 1992; Dunn and Restall, 2002; Mares, 2024). Hence, the investigations on relevant consequence relations can be naturally placed within the original goal in terms of conditionals. In this Chapter and the next ones, the focus will be on the study of relevant consequence relations, although some connections will be drawn between these and valid conditionals.

Different formulations of relevance have been explored in the literature, some of them independent from one another, others being refinements of previous versions. The general idea that motivates all of these specific notions can be informally put as follows:

> A valid argument cannot be such that its premise and conclusion are totally unrelated: they must be relevant one for the other.

Of course, this requirement is vague and can be further specified leading to different ways to understand what it means to be relevant. With no further specifications of what it means to be relevantly related, this

[1] While conditionals only have one antecedent and one consequent, arguments can involve multiple premises and multiple conclusions (or sometimes only multiple premises and one conclusion). For the moment only single premise-single conclusion arguments are mentioned as to keep the analogy with the conditional but, as it will become clear in the upcoming Chapters, multiple premises and multiple conclusions play a decisive role in the logics that will be considered.

idea can be employed to justify the most classical notion of such relation: truth preservation could be sufficient for this demand of not-being-totally-unrelated, hence classical inferences may be seen as relevant. However, since its origins, the notion of relevance was precisely thought of as a way to expect a more substantial connection between the premise and conclusion of a valid inference. As mentioned above, the development of relevance logics mainly began as a reaction to the so-called paradoxes of implication highlighted by C.I. Lewis (Lewis, 1912, Lewis, 1917, Lewis, 1918). Among them, the fact that true propositions are implied by anything and that false propositions imply any other proposition. Lewis' development of strict implication, which was meant to internalize a notion of entailment, did solve some of these issues, but not all: the so-called paradoxes of strict implication remained. This motivated the work of Anderson and Belnap, 1975 who, following the pioneering work of Church, 1951 and Ackermann, 1956, developed the logic of entailment **E** and the logic of relevance **R**, both of which succeeded in avoiding both the paradoxes of implication and the paradoxes of strict implication. These and related logics were then extensively studied by Anderson and Belnap, as well as Dunn, Meyer, Sylvan, Plumwood, Urquhart and others (see *e.g.* Dunn, 1966; Meyer, 1966; Routley and Meyer, 1973; Routley et al., 1982; Urquhart, 1972.).

The diagnosis that this literature provided for such paradoxes was precisely the fact that material and strict implication could hold between sentences that are totally irrelevant one for another. Namely, while these implications simply amount to relations between truth values or truth conditions, relevance demands a more substantial, meaningful connection, which cannot be reduced to either of these[2]. If that were the case, then any logical falsity would be relevant for any other statement and any statement would be relevant for any logical truth[3]. Among the paradigmatic examples of irrelevant inferences there are such cases as the following:

Ia. *The Moon is made of green cheese. Therefore, either it is sunny in Turin or it isn't.*

Ib. *It is sunny in Turin and it isn't. Therefore, the Moon is made of green cheese.*

As Anderson and Belnap, 1975 put it, the guiding intuition for relevance is that there must be a 'commonality of meaning' between the premise and conclusion of an argument[4]. Since truth conditions alone do not suffice to guarantee this, another component must come into play. In Examples Ia and Ib, what stands out immediately is that one side of the argument is *about* the composition of the Moon, while the other is *about* the weather in Turin. The two sides are thus concerned with completely different *topics*, which easily explains the irrelevance of both arguments.

The idea that an overlap in *subject matter* provides the missing link for the commonality of meaning mentioned above—and thus for relevance—aligns

[2]It is important to mention that this is not the only position among relevance logicians, as some consider the classical notion of truth preservation to be compatible with relevance. For instance, Read, 1988 attributes the failure of relevance to the way premises and conclusions are conjugated rather than to the definition of validity as mere truth preservation.

[3]To be more precise, this would follow from a classical treatment of logical falsehoods and truths, but may not happen in non-classical contexts. For instance Belnap-Dunn's logic **BD**, also called $\mathbf{E_{fde}}$ or simply FDE (namely, the first degree entailment fragment of **E** by Anderson and Belnap, 1975), is, in some sense, a relevant consequence relation defined as truth preservation—in a scenario where truth value gaps and gluts are admitted—which avoids this consequence.

[4]Although the focus here is on relevance logics, the attempt to align validity more closely with a connection in meaning can likewise be found in other logical systems, *e.g.* the logics that Sylvan, 2000 named 'sociative', as well as connexive logics (*e.g.* Nelson, 1930) or the proposal by Epstein, 1979.

naturally with the conception of *content* proposed by Yablo, 2014. According to this view, a statement's content consists of two components: on the one hand, as in the standard account, its truth conditions, and, on the other hand, its subject matter (this is discussed in more detail next in Section 1.3). Therefore, if this two-sided characterization of content is accepted, a necessary condition for commonality of meaning is shared subject matter. While commonality of meaning would also entail shared truth conditions[5], this may prove too stringent in the context of a consequence relation where mere preservation of truth suffices. Nevertheless, the requirement of topic overlap alone is sufficient to rule out standard examples of irrelevance—such as Examples Ia and Ib—or similar inferences that, although vacuously valid in a classical framework, lack any substantive connection between premises and conclusion.

This makes the demand of relevance in inferences tightly connected with the introduction of topic-theoretic concerns for valid inferences mentioned above. Notice that this is not in contrast with the canonical conception of validity as a formal relation holding irrespective of the particular subject matter involved, because subject matter can be formalized in several ways that are consistent with the schematic understanding of inferences. Relevant consequence relations, including all those considered throughout this work, are closed under uniform substitutions, leaving untouched the formal character of validity. In fact, the topic theoretic concerns that enter in the definition of a relevant consequence relation, even in the most refined cases, are themselves expressible as formal relations.

[5]Because under the Yablovian notion of content two statements have the same meaning if and only if they share both topic and truth conditions.

1.2 Content inclusion

As discussed above, an arguably necessary condition for relevance is subject matter overlap. Depending on how topic is modeled, there can be more or less fine-grained ways to conceive this requirement—based, for instance, on the way topics can be combined and the role, if any, played by the logical connectives.

An additional and distinct question deserves attention: is mere *commonality* of content, thus topic sharing, sufficient for relevance, or is a stronger connection between premise and conclusion required—or, at least, worth considering? This work focuses on this further possibility. In particular, while the most common option described above sees relevance as a connection of meanings such that premise and conclusion share *at least part* of their meaning, another option tries to answer the following question—that, in some sense, counts the previous case as a special one. Which inferences are such that the *whole* content of the premise stands in such relation with (even just a part of) the content of the conclusion, or vice-versa, such that the *whole* content of the conclusion holds this connection with (even just a part of) the content of the premise? This amounts to the question of which inferences are such that there is not mere overlap of subject matter, but, rather, inclusion of the same. Consider, for instance, the following examples:

IIa. *The Moon is made of green cheese. Hence, either the Moon is made of green cheese or it is raining in Turin.*

IIb. *It is raining in Turin and the Moon is made of green cheese. Hence, it is raining in Turin.*

Although such inferences do meet the requirement of meaning commonality, they fall short of satisfying one version of the stronger condition of content inclusion (either from conclusions to premises or vice-versa). In particular, in Example IIa, part of the conclusion's content exceeds the premise's, while in Example IIb, part of the premise is not relevant to the conclusion. This is not to say that, then, under *any* understanding of relevance as meaning connection, these examples too should be tagged as failures of relevance. Nevertheless, Example IIa can be recognized as a case in which the conclusion introduces something irrelevant for the premise, and Example IIb as a case in which something irrelevant for the conclusion is included in the premise—where, for both, irrelevance is understood just as previously.

One natural question is whether the same motivation which guided the development of relevance logics in the first place is at work in this stricter understanding of relevance as content inclusion. As mentioned above, the investigation on relevance logics began as a way to avoid certain paradoxes of implication which could connect totally unrelated statements. Clearly, the picture here is different since the requirement of a commonality of meaning is implied by that of meaning containment, hence what was already discarded as irrelevant in the weaker version of relevance is automatically discarded in the stronger version. Arguments as those in Examples IIa and IIb do not necessarily stand out as 'paradoxical' in the same way as Examples Ia and Ib, and in fact they do not precisely because the latter arguments and not the former have a premise and a conclusion which are about totally different topics. However, this outcome is expected: it makes sense to consider that the paradoxical character of certain implications correlates directly with the degree of irrelevance involved. Consequently, the less irrelevant two statements are to one another——or, equivalently, the stronger the relevance between them——the less unintuitive the implication connecting them will be. Nevertheless, it is still compelling to explore stricter notions of relevance once the weaker one is embraced, since content inclusion discards, in one direction, arguments in which not *all* of the conclusion's content is relevantly implied by the premise, or, in the other direction, arguments in which not *all* the premise's content is relevant for the conclusion.

Notice that, just as before, accepting the two-component characterization of content by Yablo, 2014 as comprised both of an alethic and a topical component, the failure of content inclusion in these examples can be explained accordingly. In particular, Yablo's central thesis for this purpose is the following:

> Content inclusion is implication plus subject matter inclusion. Both of these are relations in which a semantically important property is preserved: truth, in the one case, and aboutness, in the other. So the proposal can be put like this:
>
> > [The content of] B is part of [the content of[6]] A iff the inference from A to B is:
> >
> > (i) truth preserving—A implies B
> >
> > (ii) aboutness-preserving—A's subject matter includes that of B.
>
> (Yablo, 2014, p. 15)

[6]The phrases within squared brackets in citations are not in the original.

It is important to remark that, in Yablo's view, content inclusion is only characterized from right to left, so, in the case of inferences, the content of conclusions being included in that of the premises. Moreover, the requirement of subject matter preservation (from premises to conclusion) amounts to the topic of the conclusion being included in that of the premise (so right to left topic inclusion), so inclusion of topic really is in the opposite direction than truth preservation.

It seems that the other direction of relevance remains out of the picture in this context. However, Yablo's account can be extended at least in one reasonable way so as to also characterize a consequence relation as a relation of containment of content from left to right. This idea can be somehow implicitly found in some of the literature but has not been explicitly discussed yet. So, one way to phrase it is the following.

The goal is to find a notion of content inclusion which is somehow the inverse of Yablo's. Simply inverting Yablo's idea would amount to inverting points (i) and (ii) above. The result of this operation would be that, an inference from A to B is such that the content of A is included in that of B if and only if (i)^{inv} B implies A—or, reading it from left to right, the inference is falsity preserving—and (ii)^{inv} B's subject matter includes that of A—so again reading it from left-to right, the inference is non-aboutness preserving.

Although this may be a plausible move by itself, it strikes quite immediately that (i)^{inv} is in tension with the presupposition of there being an inference from A to B in the the first place. If this clause is kept as it is, then what is really taken into consideration is the inference from B to A. But, given clause (ii)^{inv}, this definition just collapses into Yablo's account of the content of the conclusion being included in that of the premise. Moreover, recall what was noted above: in Yablo's definition, topic inclusion has to go in the opposite direction with respect to the direction of preservation for the relevant alethic property.

For this reason, the simple strategy of inverting the clauses in Yablo's definition cannot work. In order to fully keep the analogy with Yablo's account of right-to-left content inclusion, preservation of the relevant alethic property has to proceed in an opposite direction with respect to that of topic inclusion, at the same time keeping the orientation of the inference as it was assumed. Therefore, in order to define a genuinely distinct notion of content inclusion from left to right, an alternative formulation has to keep clause (ii)^{inv} as it is—since topic inclusion must go left-to-right—while changing clause (i)^{inv}.

The most natural way—and probably the most conservative with respect to Yablo's original intuition—to perform this modification, is to change the just mentioned *relevant alethic property* to be preserved from B to A. Namely, given that implication can be seen equivalently from left to right as truth preservation and from right to left as falsity preservation, clause (i)^{inv} can be rephrased as *falsity* preservation from B to A—equivalently, non-falsity preservation from left to right, *i.e.* A implies B. Hence, in analogy with the characterization by Yablo for right-to-left content inclusion, this same relation in the other direction can be taken to be preservation of falsity from the conclusion to the premise and preservation of subject matter in the same direction—namely, just as before, inclusion of topic in the opposite direction[7]. In a simplified scheme:

[7]Notice that another—probably less cumbersome—way to tackle the issue is to recognize that in Yablo's definition, truth preservation leans itself to several potentially

The content of A is part of the content of B iff the inference from A to B is:

(i)^{inv} falsity preserving from B to A—B is implied by A

(ii)^{inv} (reversed) aboutness-preserving—B's subject matter includes that of A.

Therefore, just as in the previous section failure of relevance in the sense of meaning commonality could be explained as failure of subject matter overlap, Example IIa and Example IIb can be viewed as inferences where content inclusion fails not as a consequence of some alethic property not being preserved, but once more because of the topical constraint.

Before proceeding, it is worth adding some historical notes on content inclusion logics in order to locate with more precision the aim of this inquiry. The idea of valid arguments carrying some containment relation can be found in many authors since the origins of logic. As noted in Ferguson, 2017, even Aristotelian syllogistic can enter in this picture as well as some positions attributed to the Stoics that view logical inference in general as a process of *analysis*. This seems to be the case, for instance, for the fourth interpretation of the conditional discussed by Sextus Empiricus in his *Outlines of Pyrrhonism* according to which "a true conditional is one whose consequent is potentially included in the antecedent."(PH 2.112; cf. 35B p.209, translated by Long and Sedley, 1987). Crucially, this reading can

distinct specifications. Namely, truth may as well be intended as non-falsity, and falsity as non-truth. Therefore, distinguishing between preservation of truth and preservation of non-falsity may just be a distinction in terminological choices, not necessarily leading to distinct notions of implication. And even when such a distinction happens, for instance when truth and falsity are not exhaustive, then there seems to be no specific reason why Yablo's definition would only admit one kind of specification for truth preservation and not others.

be explicitly found in the Kantian characterization of *analytic* judgments, distinguishing them from *synthetic* judgments. In the author's own words:

> Either the predicate B belongs to the subject A as something that is (covertly) contained in this concept A; or B lies entirely outside the concept A, though to be sure it stands in connection with it. [...] One could also call the former judgments of clarification and the latter judgments of amplification, since through the predicate the former do not add anything to the concept of the subject, but only break it up by means of analysis into its component concepts, which were already thought in it (though confusedly); while the latter, on the contrary, add to the concept of the subject a predicate that was not thought in it at all, and could not have been extracted from it though any analysis. (Kant, 1998, p.130)

Kant's appeal to the analytic process of 'unpacking' and the synthetic procedure of 'expanding' a concept was thought of within a specific philosophical and foundational framework and the scope of this characterization were judgments, rather than inferences (or conditionals). However, it is clear that both cases involve a kind of reasoning which put in connection different contents, where one is contained in the other. The most important feature of Kant's account, for the purposes of this work, is the fact that concepts—rather, contents—are taken as the primitive notion upon which the relation of analyticity or syntheticity is defined. As Ferguson, 2017 notices, this is in contrast with what Gemes, 1994 calls the 'traditional concept of content', which instead defines the content of some statement as the set of statements which logically follow from it.

What emerges from this picture is that there may be two ways to talk about content inclusion, or relevance in general. According to one view, content inclusion is somehow already thought in terms of closure under logical consequence. Therefore, it heavily depends on the underlying system, as choosing one logic over the other settles the question of what statements are part of the content of some other statement and thus straightforwardly answers what inferences are relevant (that is, all the valid inferences of that system). According to the other, taking some notion of content, hence of subject matter, as primitive, allows to meaningfully ask what inferences are relevant in the preferred sense—namely, displaying commonality or containment of contents—and whether a logical system allows for violations of relevance. The investigation carried out here is in line with the second view. First, several ways to conceive subject matter and subject matter inclusion, hence content inclusion, are considered. Then, assuming one of these definitions, it is shown which logical systems can be seen as content inclusion logics, whether some subsets of the valid inferences in a logic are relevant in our sense and whether some violations are allowed or not. The next step is to examine some of the ways in which the subject matter and content of a statement can be defined and modeled formally. Subsequently, various options for the characterization of topic and content inclusion will be considered.

1.3 Subject Matter and Truth Conditions

A formal account of subject matter is the first step towards a precise account of relevance. Given the considerable amount of such accounts in the literature, the focus will be limited to a subset of proposals that share a distinct syntactic flavor, in a sense which will be clarified soon.

First, it is important to bear in mind that throughout this work only a propositional language is considered, so there is a limitation to the fine-grainedness that can be achieved in the formalization of arguments and, consequently, in the formal account of the subject matter of sentences. This limitation lies in the fact that, once two sentences are formalized as *e.g.* p and q it is not possible to, so to say, look inside them and identify a common term, hence a common reference, that could be interpreted as a shared subject matter or content. So, given that the formal investigation starts with an already formalized language, the issue here is not what constitutes the subject matter of a sentence, but rather what formal feature can be identified as such in a formalized language. The syntactic spirit mentioned earlier lies precisely in this intuition: what can intuitively be the bearer of subject matter and what could make transformations on it should be identified in the syntax of a formal language.

With this in mind, Yablo's (Yablo, 2014) main position on *aboutness* of declarative statements can now be reviewed, in order to provide a clearer philosophical background for the formal frameworks that follow. In fact, the usefulness of this account in the context of relevant logics does not *per se* serve as a robust motivation for it. Independent reasons are needed to take seriously this two-component view of content. As a reference, the introduction of Yablo's positions follows Berto, 2022, further discussed and defended in Hawke et al., 2024.

The first guiding intuition is the following: declarative sentences allow to express something which can be true or false about certain topics. In the examples above, ‘It is raining in Turin’ makes a claim that can be evaluated as true or false about Turin or the weather in Turin, while ‘The Moon is made of green cheese’ says something false concerning the Moon and its composition and specifically concerning the cheese that forms it. Hence, this intuition can be explained as the fact that what is expressed by a statement φ, call it a proposition or its content, is a pair $C = \langle A, T \rangle$ where A refers to the truth conditions of φ and thus determines its alethic status, while T refers to the topic(s) φ is about, determining its subject matter. The natural question then is the following: is one of these, A or T, reducible to the other? If the answer turns out to be positive, then it doesn’t make sense to talk independently of truth conditions and subject matter. In particular, if the latter can be reduced to the former, the explanation of relevance failures provided above would ultimately require appeal to alethic considerations.

The case in which A and T collapse is called in Berto, 2022 the one-component account (1C) of propositional content, and a semantic framework which implies it is said to be a one-component semantics. Instead, keeping them apart amounts to the two-component account that was already considered above, in Berto, 2022 called (2C), implied by two-component semantics. More precisely, (1C) can be taken as the thesis that either there is a function mapping T into A for every content C, or vice versa that truth conditions can be mapped into topics; instead (2C) is the negation of this disjunctive claim.

Interestingly, there are examples of hyperintensional one-component semantics. For instance, in the approach by Fine (*e.g.* Fine, 2016b, Fine, 2017, Fine, 2020) given a statement, A is an ordered pair of verifiers (truthmakers) and falsifiers (falsemakers), while T is defined as the fusion of all such truthmakers and falsemakers. The fact that hyperintensionality can also be achieved without embracing a two-component semantics suggests that, at least on the technical side, there is nothing essentially in favor of adopting a two-component account in order to achieve relevance. The matter should then be assessed in terms of conceptual advantages. This issue can be considered with the following remarks. Firstly, it seems that even conceptually there is no definitive argument for one approach or the other. The reason for this claim is that assessing which account better captures our intuitions is somewhat circular. For instance, in order to support (2C) one would need to find examples of sentences with different meaning but sharing either the same alethic status (as in Hawke et al., 2024) or being about the same topic, thereby showing that meaning cannot be reduced solely to either factor. However, the evaluation of both statuses depends on one's assumptions on the ascriptions of truth or falsity as well as of aboutness and, in particular, on the (in)dependence of such assignments. Secondly, although (1C) seems to be more conceptually parsimonious in that it maintains that truth conditions or subject matter are after all reducible one to the other, in order to entail a hyperintensional notion of content which aligns with many of the usual intuitions in these respects, it also needs to depart to some extent from the most standard notions of truth conditions or aboutness, as it is also discussed in Berto, 2022. For instance, if truth conditions were reduced to topics, then explaining the meaning difference between *e.g.* 'It is raining in Turin' and its negation, 'It is not raining in Turin', would amount to arguing that such statements differ in their subject matter. Although the fact that negation can trans-form the topic of a statement is taken seriously in some works on relevance logics, this idea is far from uncontroversial and often rejected. Even the other direction of the reduction for (1C), namely the case in which subject matter is reduced to truth conditions, seems to encounter similar problems. For instance, explaining in alethic terms the meaning difference between 'It is sunny in Turin or it isn't' and '2+2=4', needs to appeal either to entities such as impossible worlds, or incomplete/inconsistent states, or to other alethic statuses than mere truth and falsity. In contrast, (2C) is more neutral in this respect because, although it does add one extra irreducible feature to statements' contents, it does not necessarily come with other departures from more or less standard alethic or topical assumptions (see Chapter 4). The meaning difference between the previous examples can be spelled out simply in terms of distinct truth conditions in the first case, in terms of topical difference in the second case (as it was also mentioned at the beginning of the Chapter).

Therefore, although Yablo's two-component characterization of content, as well as that of content inclusion, may not have specific advantages in the technical applications when pursuing relevance, its conceptual neutrality does come with interesting upshots. It will be argued that such neutrality has its benefits also because it allows for semantic characterizations that are very general and possibly open to other interpretations.

1.4 Modeling Subject Matter

After having laid down the philosophical background, various approaches to formally modeling subject matter can be introduced. As discussed above, the question of what it means for the premise and conclusion of an inference to share some content or for one of the two to include the content of the other, contains the question of what it means for subject matter to be shared or included. In fact, embracing Yablo's view we know that content is given both by truth conditions and subject matter, hence content sharing or inclusion entails topic overlap or containment, respectively. How is it possible to assess whether this necessary requirement is met? In this respect it is crucial to fix one way to account for subject matter in a formal language, so that this evaluation can be carried out for any formal system over the same language.

It was mentioned above that the goal is to find a characterization of subject matter which can be read off from the syntax of the language. Now, what could be a good candidate in a formal language to carry the subject matter of a formula? It seems that the only plausible candidates are, on the one hand, propositional letters and, on the other hand, logical operators. In the case of atomic formulas though, as no operator occurs in them, the only bearer of subject matter can be the propositional variable constituting the formula itself, so the role of logical operators, if any, must come into play only when considering complex formulas. So in the base case, the bearer of the topic expressed by an atomic sentence p is p itself, making it possible to identify each propositional variable with a different topic.

Before moving on, it is useful to pause and consider one possible objection to this very first stance. Consider, for instance, two sentences as 'The wardrobe is old' and 'The armoire is old'. Given that they're not the same sentence, they could be naively formalized with two different variables, say p and q respectively. However, this choice is clearly unfortunate as the topic as well as the truth conditions (hence the meaning) of these sentences should be counted as being exactly the same, while with this method there would be a split of two contents which can simply be expressed using different terms, just as ‘wardrobe’ and ‘armoire’ refer to the same object. There are at least two ways to reply to this objection, both showing that it does not constitute a real problem. One is to reject the formalization itself of such statements as distinct propositional letters. Namely, an adequate formalization should take into account whether some sentence is synonymous with another and therefore assign the same variable in the first place if they are synonymous. The other reply, provided by Hornischer, 2020, can be rephrased as follows. Even if different atoms were used to formalize sentences with the same meaning, it is possible to assume any preferred theory that fixes which atomic sentences in natural language are synonymous (so for the example above, a theory which counts ‘wardrobe’ and ‘armoire’ as referring to the same object, hence the two sentences expressing the same proposition) and a $\approx$-equivalence relation can be built so that whenever such theory counts two variables p and q to represent synonymous statements, it is possible to stipulate that $p \approx q$. Then, a new set of variables can be added, call them p_0, p_1, p_2... each of which standing for a single $\approx$-equivalence class, and one could just work with these.

Both approaches offer a straightforward solution to some of the limitations of a purely syntactic characterization, though it’s important to acknowledge that some other weaknesses persist as they both only allow to identify statements that are synonyms. For instance, they cannot account for situations in which two logically atomic statements in natural language share a topic but differ in their content—*e.g.* 'the wardrobe is old' and 'the wardrobe is new', both being *partly* about the wardrobe but differing in the overall subject matter and truth conditions. In such cases, both strategies would assign them different variables, failing to recognize their shared subject matter. However, at least cases as the one just mentioned seem to depend on the structural coarse-grainedness of a propositional language. Moreover, notice that the example considered could be a problem if relevance were characterized as commonality of meaning, hence as topic overlap, but not if, as it is done here, the stronger notion of content inclusion, hence topic containment, is embraced—as it would in fact fail in both directions for the statements above. It is not hard though to find other examples of content inclusion which cannot be seen with these methods. For instance, the proposition expressed by 'Turin is in Europe' is in some sense part of that expressed by 'Turin is in Piedmont'. Although working with a first-order language may somehow account for a commonality of meaning (since Turin appears in both statements), by itself it cannot account for the fact that being in Europe is somehow analytically contained in being in Piedmont. Clearly, another strategy is needed for this which accommodates a higher level of fine-grainedness.

Having considered the limitations of a purely syntactic formal account of subject matter, possibly solvable moving to a more fine-grained language, it is now possible to take the next step, provided that in the atomic case the topic of a statement simply coincides with the propositional variable used to formalize it. The next step concerns logical operators and, in particular, the question of whether such operators can be, employing the terminology introduced in Ferguson, 2023b, *topic transformative*—that is, whether they can induce a shift in the subject matter of the formula(s) within their scope—or whether they are instead *topic transparent*. The literature on topic-sensitive logical frameworks tends to favor the latter view (*e.g.* Berto, 2022, Perry, 1993, Fine, 2020, Epstein, 1990, Beall, 2016, Paoli et al., 2021, Bonzio et al., 2022), although there are significant contributions adopting the former position as well (*e.g.* Angell, 1989, Fine, 2016a, Ferguson and Logan, 2023, Szmuc and Rubin, 2022, Randriamahazaka, 2022, 2024). Both options will be examined.

1.5 Topic-transparent Logical Operators

First of all, it is necessary to determine which logical operators are to be considered and accounted for. The propositional language under consideration may include negation, conjunction, and disjunction. As for the conditional, it will largely be left aside, since, as noted at the beginning of this chapter, the focus is on a notion of relevance for the entailment relation. There is also a technical reason for this that will be discussed later in Section 2.1, namely the impossibility result by Ferguson, 2017, ch.1, p. 7 which shows how it is not possible to both have an entailment and a conditional which are relevant. Hence, in order to maintain a relevant entailment, the conditional may simply be taken as the material one, defined in the usual manner through the other connectives. Some remarks on it will nevertheless be made in the discussion of topic-transformative operators.

The question is how to determine the topic of a complex formula whose main operator is one of the three mentioned above. As Berto, 2022 puts it:

> 'Jane is not a lawyer' is exactly about what 'Jane is a lawyer' is about. 'John is tall and handsome' and 'John is tall or handsome' are both about the same topic: the height and looks of John. There is no obvious candidate for a topic that is systematically introduced by a negation, a conjunction, or a disjunction. (Berto, 2022, p. 32)

It is worth pausing briefly to emphasize the use of the word 'systematic'. The main point, as in Hornischer, 2020, is that, if any of these operators were to introduce a topic, this topic should be the same each time since the meaning of logical operators, if they have any, shouldn't arbitrarily change. So what could be the topic of negation? If there were any, then any negated sentences such as 'Turin is not rainy' and 'Jane is not a lawyer' would happen to share at least some subject matter, namely that of negation. The same argument would work for conjunction and disjunction: the subject matter of 'Turin is rainy and (or) the Moon is made of green cheese' on the one hand, and that of 'Jane is a lawyer and (or) John is tall' on the other hand, would also overlap as both statements would both partly be about the topic of conjunction (disjunction). Clearly, these conclusions cannot be reasonably accepted, therefore this way of putting the problem only favors a treatment of such operators as topic-transparent.

To sum up, combining this view with the syntactic characterization of the subject matter of a logically atomic statement, the topic of any statement formalized by a formula φ ends up being nothing but the set of variables occurring in φ, indicated with $Var(\varphi)$[8]. More will be said on

[8]Notice that the implicit assumption is that the overall subject matter of a statement

the formalities of the so-obtained structure of topics in the next Chapter. From now on, this way of characterizing the subject matter of a statement will be called TTO—standing for Topic Transparent Operators:

$$\textbf{TTO}\text{: Topic of } \varphi \ = \ Var(\varphi)$$

1.6 Topic-Transformative Logical Operators

It was noted above that the view of negation, conjunction or disjunction as being topic-transformative, in the sense of inherently carrying a topic, is not easily tenable. There can be nevertheless other ways to defend topic-transformativeness of some operators. In the literature there is not much about conjunction and disjunction being so (*e.g.* Correia, 2016), while there is definitely more on negation. Before outlining some of the reasons for and against a topic-transformative view of negation, it is useful to briefly add some remarks on another operator which is instead mostly taken to be so, namely the intensional conditional—as well as intensional operators in general—and the relevant conditional in particular. Ferguson, 2023a explores three ways in which such operator may be topic-transformative. First, a conditional could be *ampliative* with respect to the topic of its antecedent and consequent, thus adding its own subject matter—namely, the relation of implication carried by it. This account, though, seems to encounter the problem discussed above to defend topic-transparency. Second, in the case of some counterfactuals, the topic of the conditional could be *incommensurable* with respect of the union of the subject matter of its antecedent and consequent. Third, a conditional could be *order sensitive*.

is the *set* of its subtopics, namely that the order of subtopics and their repetition does not count. This assumption may be challenged, but this is left for future investigation.

In fact, according to the author, some accounts of conditionals such as the restrictor view by Kratzer, 1991, can be interpreted as implying that reversing antecedent and consequent results in a shift of topic, given that the antecedent determines what the focus is restricted to.

Moving to the case of negation, there seem to be in general less compelling intuitions with respect to it being topic transformative. As was already remarked, supporting the idea that negation is topic-transformative in the sense of carrying itself its own topic, hence an *ampliative* account of negation, faces several limitations. What, then, might constitute a reasonable perspective on this matter? The following is a well-known example frequently cited in the literature in support of such a position:

> For example, '(Jo died and Jo did not die and Flo wept)' does not mean the same as '(Jo died and Flo did not weep and Flo wept)'; for the first contains a false and inconsistent statement about Jo though the second does not, while the second contains a false and inconsistent statement about Flo though the first does not. How can two sentences mean the same thing if one contains a false and inconsistent statement about an individual while the other does not? A syntactical condition which will rule out such cases can be formulated using a distinction by Herbrand between 'positive' and 'negative' occurrences of a variable in a schema.
>
> (Angell, 1989, p. 121-122)

As it is noted in Ferguson and Logan, 2023, the fact that Angell's example supports the idea that negation is topic-transformative is dependent on the assumption that contradictions have equivalent truth conditions, which is in tension with the mainstream literature on relevance logics[9]. However, the authors themselves acknowledge that also assuming the opposite, namely that contradictions do not necessarily share the same truth conditions, could rule out several ways in which topics can be assigned. For instance, in the account by Fine, 2016a, the subject matter of a statement is identified with some closure of the set of its verifiers, and in the case of a statement and its negation these sets, hence their topics, may differ.

Angell's quote may also suggest another way to track how subject matter is altered by negation, that is, with a change of *polarity*. Namely, topics may be presented positively or negatively just as variable occurrences may be, depending on the number of negations, even or odd, under which they occur. So what negation performs is not an addition of its own subject matter, rather a shift in the presentation of a topic. 'Turin is rainy' and 'Turin is not rainy' do share the same topic though it is, in the first case, presented positively, while, in the second case, negatively. However, whether the way of *presenting* a topic should be taken seriously or not in the actual consideration of *which* is the subject matter of a statement, is not that straightforward. It seems though that some intuitions on other cases which do not involve negation may be applied to this case, to support in general the idea that the *presentation* of a subject matter makes a substantial difference. Such intuitions can be borrowed from Yablo:

> Why is MAN BITES DOG a better headline than DOG BITES MAN? One thing we can certainly say is that it is on a more

[9]For instance, in the four valued setting of $\mathbf{E_{fde}}$ contradictions do not necessarily have the same truth conditions. It is worth anticipating that this case is particularly important in the case of Angell's proposal because of the tight connection between Angell's logic of analytic containment **AC**, that will be discussed in detail, and $\mathbf{E_{fde}}$.

interesting topic. A more interesting topic is a different topic.

Yablo, 2014, p.24

A topic being interesting or not seems to depend on its *presentation*. So, taking these intuitions seriously, may imply that negation can be topic-transformative: just as “Man bites dog” differs in topic from “Dog bites man”, so too “Man does not bite a dog” should differ in subject matter from “Man bites dog.” for the same reason[10]. Notice that this kind of argument cannot work for *every* case as the same shift in interest, hence in topic, is not a systematic phenomenon—for instance, it doesn’t seem in general that statements such as ‘Turin is rainy’ and ‘Turin is not rainy’ are about different topics because one is more interesting than the other.

Connecting to the view of negation as shifting between polarities, there is at least another way to see it as topic-transformative. Rather than assuming that topics may have different polarities as a consequence of them being presented differently—a view which, as it was mentioned above, would need to be further justified—it seems more natural to distinguish directly between positive and negative topics. For example, consider again the pair ‘Turin is rainy” and ‘Turin is not rainy.” One might argue that the former is about what the weather in Turin *is*, while the second is about what the weather in Turin *is not*. On this account, polarity is always tied to the subject matter of a statement. Although this seems relatively intuitive, there’s at least three issues to consider. Firstly, this proposal entails that

[10]Notice that here and in Yablo’s example one may object that there is only a difference in truth conditions. However, this objection can somehow be resisted a follows. Consider the statements ‘Man bites dog or, either it rains or it doesn’t’ and ‘Man does not bite dog or, either it rains or it doesn’t’. They are classically equivalent but may be counted as having a different topic, whose difference can only be imputed to the first disjunct of each case. However, with this approach it is not just as clear as before whether one statement is more *interesting* than the other.

negation is always topic-transformative. While this is acceptable in principle, as it will be shown in more detail a relevant subfamily of containment logics only supports a weaker position which allows for a statement and its negation to possibly be on the same topic as well as differing in this respect. Therefore, this stronger view could not account for the notion of content inclusion in these systems. Secondly, an implicit assumption is that there are only two polarities. This would mean, for instance, that 'Turin is rainy" and 'It is not the case that Turin is not rainy" share the same subject matter. Yet, just as certain logics distinguish a statement from its double negation in terms of truth-conditions, one may also ask whether a statement and its double negation are about the same topic. Answering to this question is not straightforward and reasons are needed to justify either possibility. Thirdly, a possible worry is that this view sets up a slippery slope toward conflating subject matter with syntax. Without further constraints, one might be led to identify the subject matter of a statement with the statement itself, not only possibly admitting as many polarities as the number of negations of a statement, but also drawing distinctions between, say, conjunctive and disjunctive subject matter. While the suggested view does not commit to this collapse, additional arguments are required to prevent it.

To conclude, several ways in which the literature conceives of negation as topic-transformative have been examined, alongside some alternative proposals. However, at this stage none of these accounts appears to be sufficiently well supported, either conceptually or intuitively. Therefore, further work is required in order to develop a more robust and well-grounded understanding of negation in its alleged topic-transformative role.

This Section can be closed with the following remark. While in the case of topic-transparent operators the topic of a formula can simply be represented as the set of variables appearing in it, the topic-transformative view of negation leads to a different outcome. Building on the position advanced by Angell, 1989—a central reference in formal accounts of relevance that regard negation as topic-transformative—– and combining it with the syntactic characterization of the topic of atomic statements, the subject matter of a formula φ can be represented as a pair: the set of variables occurring positively in φ and the set of those occurring negatively[11]. These sets are denoted by $Var^{+}(\varphi)$ and $Var^{-}(\varphi)$, respectively—a more precise definition will be given in the next Chapter, Section 2.2. This way of characterizing the subject matter of a statement will be called from now on TTN—standing for Topic Trasformative Negation:

$$\textbf{TTN}\text{: Topic of } \varphi \; = \; \langle Var^{+}(\varphi), Var^{-}(\varphi) \rangle$$

[11]This choice unveils a crucial assumption mentioned above, namely that a sentence and its double negation are always about the same topic.

Chapter 2

Logics of Variable Inclusion

Just as the broad family of relevance logics includes systems that accept only relevant inferences (or conditionals), the stricter notion of content inclusion restricts the field to the so-called content inclusion logics, also known as variable inclusion logics or simply containment logics. The main link between containment logics and the two-sided view of content is given by the characterization of their logical consequence as preserving, in some sense, both truth and topic, thus formally capturing the Yablovian thesis. Building on the preceding informal notions, this chapter formally defines content (and topic) inclusion, introduces basic variable-inclusion logics, and characterizes their valid inferences in terms of content-inclusion requirements. It also briefly reviews alternative interpretations in the literature that provide independent motivation for these systems and related ones.

2.1 Subject Matter Inclusion

Both TTO and TTN are at the basis of various ways to represent topic overlap and topic inclusion, the former being a necessary condition for many relevance logics in which inferences are meant to capture some commonality of meaning, the latter being a stronger requirement needed for inferences which encode some relation of content inclusion.

Before considering several ways to define inclusion of subject matter, it is worth mentioning the syntactic principle that, under TTO, readily captures topic overlap. This principle is central for most relevance logics and is commonly known in this literature as the Variable Sharing Property, or simply VSP. In the words of its first proponents:

> Informal discussions of implication or entailment have frequently demanded "relevance" of A to B as a necessary condition for the truth of $A \to B$, where relevance is now construed as involving some "meaning content" common to both A and B. [...] A formal condition for "common meaning content" becomes almost obvious once we note that commonality of meaning in propositional logic is carried by commonality of propositional variables. So we propose a *necessary*, but by no means sufficient, condition for the relevance of A to B [...] that A and B must share a variable.
>
> (Anderson and Belnap, 1975, §5.1.2)

While this quote spells out VSP for conditionals, the same property for inferences can be defined as follows (Γ and Δ stand for sets of formulas, which will be formally defined later):

VSP: An inference with premises Γ and conclusions Δ satisfies the Variable Sharing Property if and only if $Var(\Gamma) \bigcap Var(\Delta) \neq \varnothing$

Given these definitions, a logic is said to satisfy VSP if and only if all of its valid inferences/conditionals satisfy the respective version of VSP. With this formal notion at hand, it is now easy to see why Examples Ia and Ib fail in a logic satisfying VSP: the former can be formalized as p implying $q \vee \neg q$ and the latter as $p \wedge \neg p$ implying q, but neither argument is such that premise and conclusion share at least some propositional variable.

While this principle is widely considered to be the weakest necessary condition for a logic to be counted as a relevance logic, VSP has been object of many refinements. For instance, in Anderson and Belnap, 1975 it is also proved that some relevance logics satisfy another proviso, known as the Strong Variable Sharing Property. At the level of inferences and in a language containing a negation and no conditional, this can be seen as a version of VSP based on TTN, namely that in an inference with premises Γ and conclusions Δ, these share at least one variable occurring with the same polarity, *i.e.* $Var^{+}(\Gamma) \bigcap Var^{+}(\Delta) \neq \varnothing$ or $Var^{-}(\Gamma) \bigcap Var^{-}(\Delta) \neq \varnothing$[1]

The first formulation of a syntactic notion which strengthens VSP in terms of containment—for conditionals—can be found in Parry, 1968, in which the Proscriptive Principle is introduced:

> In general, for analytic implication, as I call it, a formula will not be valid if it permits the introduction of any term whatsoever in the consequent regardless of the antecedent [...] Hence the basic principle of analytic implication: No formula with analytic implication as main relation holds universally if it has a free variable occurring in the consequent but not the antecedent. I call this the Proscriptive Principle.

[1]For a revision of many such strengthenings of VSP and their relation to the topic transformative nature of negation and the conditional, see Ferguson and Logan, 2023.

(Parry, 1968, p.151)

Interestingly, the idea that in an implication the content of the consequent has to contained in that of the antecedent, can also be found earlier in Ackermann, 1956, which inspired Anderson and Belnap to develop their logic **R**, although it did not result in a conditional complying with Parry's proscription.

> The rigorous implication, represented as $A \to B$, is intended to convey that there is a logical connection between A and B. It signifies that the content of B is part of the content of A, or however one might express it.
>
> (Ackermann, n.d. p.5)

As noted in the previous chapter, topic modeling is independent of the definition of relevance. Thus, just as relevance as commonality of meaning can be captured by increasingly strict syntactic filters on topic overlap, the same holds when relevance is understood in terms of content inclusion. Since topic inclusion is not symmetric, unlike overlap, each syntactic criterion must be considered in both left-to-right and right-to-left directions, each constituting a necessary condition for the corresponding form of content inclusion.

Starting from the notions of topic inclusion in an argument based on TTO, the most straightforward formulations of such requirements holding from conclusions to premises or vice-versa, respectively, are the following—the first being a formalized version of Parry's Proscriptive Principle for inferences, the latter being its dual version[2]:

[2] It was also called *converse Parry property* in Došen, 1978, *regressive analyticity* in Paoli, 1992 or *reverse containment* (Ciuni et al., 2018).

RVI: An inference with premises Γ and conclusions Δ satisfies *right variable inclusion* if and only if the set of propositional variables of Δ is included in that of Γ.

LVI: An inference with premises Γ and conclusions Δ satisfies *left variable inclusion* if and only if the set of propositional variables of Γ is included in that of Δ.

Logical systems whose valid inferences satisfy some form of **RVI** are called in Ferguson, 2017 $\vdash$-*Parry logics*, while those satisfying **LVI** are named $\vdash$-*Dual Parry logics* in Szmuc, 2019. Either way, forms of variable inclusion which fall under these and more generally whose associated interpretation as topic inclusion is based on **TTO**, in the following may be referred to as *regular*, as opposed to those which are based on **TTN**, that will be described as *balanced*.

The *prima facie* formal characterizations of regular topic inclusion are simple Right or Left set-theoretic Variable Inclusion, hereafter called Universal Variable Inclusion:

RVI$_\forall$: An inference with premises Γ and conclusion Δ, satisfies Right Universal Variable Inclusion (RVI$_\forall$) if and only if, for every $\Delta' \subseteq \Delta$, $Var(\Delta') \subseteq Var(\Gamma)$, or, equivalently, if and only if $Var(\Delta) \subseteq Var(\Gamma)$.

LVI$_\forall$: An inference with premises Γ and conclusion Δ, satisfies Left Universal Variable Inclusion (LVI$_\subseteq$) if and only if, for every $\Gamma' \subseteq \Gamma$, $Var(\Gamma') \subseteq Var(\Delta)$, or, equivalently, if and only if $Var(\Gamma) \subseteq Var(\Delta)$.

However, topic inclusion can also be characterized in ways that depart from the simple versions above, for reasons that apply in at least some cases. One reason has to do with the fact that containment logics which only validate arguments satisfying $\mathbf{LVI}_\forall$ or $\mathbf{RVI}_\forall$, may not be structural. This will be shown later, though it is worth anticipating that even if by itself this is not an issue—actually, it will be argued that it is a virtue in some sense—structural consequence relations generally allow for a more standard technical study and they are closer to the most traditional understanding of what logical consequence, hence logic, is. Another reason is that neither $\mathbf{LVI}_\forall$ nor $\mathbf{RVI}_\forall$ imply VSP. In fact, when the set of premises is empty, the inclusion holds vacuously and, dually, the same happens in the case where the set of conclusions is empty. While this may be fine if the goal is simply to characterize topic inclusion, it may be important to consider if one takes seriously VSP as the minimal necessary requirement for a system to be counted as a relevance logic and, at the same time, wants to locate content inclusion logics within this family. For these reasons, the following criteria of variable inclusion with some constraint on the non-emptiness of conclusions or premises are to be considered as well:

$\mathbf{RVI}_\exists$: An inference with premises Γ and conclusions Δ, satisfies Existential (non-empty) Right Variable Inclusion ($\text{RVI}_\exists$) if and only if, for some non-empty $\Delta' \subseteq \Delta$ implied by Γ, $Var(\Delta') \subseteq Var(\Gamma)$[3]

$\mathbf{LVI}_\exists$: An inference with premises Γ and conclusions Δ, satisfies Left Existential (non-empty) Variable Inclusion ($\text{RVI}_\exists$) if and only if, for some non-empty $\Gamma' \subseteq \Gamma$ implying Δ, $Var(\Gamma') \subseteq Var(\Delta)$

$\mathbf{RVI}_{\forall\varnothing}$: An inference with premises Γ and conclusions Δ, satisfies Right Universal Non-Empty Variable Inclusion ($\text{RVI}_{\forall\varnothing}$) if and only if Δ is

[3]Notice that considering the existential version of the variable inclusion claim without the non-eptyness constraint does not make sense given that it would always be trivially satisfied by the empty set.

non-empty and for every non-empty $\Delta' \subseteq \Delta$, $Var(\Delta') \subseteq Var(\Gamma)$, or, equivalently, $Var(\Delta) \subseteq Var(\Gamma)$ and $\Delta \neq \varnothing$

LVI$_{\forall\varnothing}$: An inference with premises Γ and conclusions Δ, satisfies Left Universal Non-Empty Variable Inclusion (LVI$_{\forall\varnothing}$) if and only if Γ is non-empty and for every non-empty $\Gamma' \subseteq \Gamma$, $Var(\Gamma') \subseteq Var(\Delta)$, or, equivalently, $Var(\Gamma) \subseteq Var(\Delta)$ and $\Gamma \neq \varnothing$

As anticipated, these regular versions of non-empty variable inclusion, hence, under TTO, non-vacuous topic inclusion, can properly be seen as strengthenings of VSP:

Fact 2.1.1. Each of RVI$_\exists$, LVI$_\exists$, RVI$_{\forall\varnothing}$ and RVI$_{\forall\varnothing}$ separately imply VSP, while neither RVI$_\forall$ nor LVI$_\forall$ do.

Notice that this result connects to the remark in the first chapter of Ferguson, 2017, which states an impossibility result about the relation between consequence relations complying with Parry's proscriptive principle and conditionals satisfying the same—namely, that no system can non-vacuously satisfy this principle for both. In fact, recall that relevance, including Parry's account, was initially thought of as a property of valid conditionals. But the fact above can be seen as showing that whichever notion of relevance implying VSP at the level of inferences, will also be such that no theorems—not even for a conditional connective, if it were part of the language—could hold, as empty premises and non-empty conclusions would imply the failure of VSP. For this reason, VSP or Parry's proscriptive principle, as well as its dual, could only hold vacuously for valid conditionals, as there would be none. Notice also that although RVI$_\forall$ does not imply VSP, it also does not allow for theorems in a system as the containment principle from right to left would also fail in the case in which premises are empty. While this can be seen as showing that there is an incompatibility between the approaches to relevance which maintain the focus on conditionals and those that shift this focus to inferences, the possibility of an interaction of the two remains open for systems complying with $\mathrm{LVI}_{\forall}$.

Moreover, it is easy to check that the Non-Empty Universal Variable Inclusion proviso implies both (non-empty) Existential Variable Inclusion and Universal Variable Inclusion.

Fact 2.1.2. $\mathrm{RVI}_{\forall\varnothing}$ implies both $\mathrm{RVI}_{\exists}$ and $\mathrm{RVI}_{\forall}$. Similarly, $\mathrm{LVI}_{\forall\varnothing}$ implies both $\mathrm{LVI}_{\exists}$ and $\mathrm{LVI}_{\forall}$.

Finally, with some set-theoretic basic notions at hand, it also straightforward to check that the converse implication holds in both cases.

Fact 2.1.3. $\mathrm{RVI}_{\exists}$ and $\mathrm{RVI}_{\forall}$ jointly imply $\mathrm{RVI}_{\forall\varnothing}$. Similarly, $\mathrm{LVI}_{\exists}$ and $\mathrm{LVI}_{\forall}$ jointly imply $\mathrm{LVI}_{\forall\varnothing}$.

This shows both that $\mathrm{RVI}_{\forall\varnothing}$ can be obtained as the intersection of $\mathrm{RVI}_{\exists}$ and $\mathrm{RVI}_{\forall}$ and that $\mathrm{LVI}_{\forall\varnothing}$ coincides with the intersection of $\mathrm{LVI}_{\exists}$ and $\mathrm{LVI}_{\forall}$[4].

Each of the previous formulations of variable inclusion can also be strengthened as to capture a notion of topic inclusion which is based on TTN, hence paying attention to the polarity of the variables occurring in a formula. In order to make explicit that some variable inclusion constraint is of the *balanced* kind (also named *refined*—*e.g.* in Szmuc and Rubin, 2022—or *signed*—*e.g.* in Randriamahazaka, 2022), a supraindex $^{\pm}$ will be

[4]Namely, the set of inferences complying with $\mathrm{RVI}_{\forall\varnothing}$ is the intersection of the set of the inferences that satisfy $\mathrm{RVI}_{\exists}$ on the one hand and the set of inferences that satisfy $\mathrm{RVI}_{\forall}$ on the other. The analogous holds for the left-to-right versions of such constraints.

explicitly added to each of them. For instance, $\mathrm{RVI}^{\pm}_{\exists}$ and $\mathrm{LVI}^{\pm}_{\exists}$ correspond to the following:

$\mathbf{RVI}_{\exists^{\pm}}$: An inference with premises Γ and conclusion Δ, satisfies Balanced Existential (non-empty) Right Variable Inclusion ($\mathrm{RVI}_{\exists}$) if and only if, for some non-empty $\Delta' \subseteq \Delta$ implied by Γ, $Var^{+}(\Delta') \subseteq Var^{+}(\Gamma)$ and $Var^{-}(\Delta') \subseteq Var^{-}(\Gamma)$

$\mathbf{LVI}^{\pm}_{\exists}$: An inference with premises Γ and conclusion Δ, satisfies Balanced Left Existential (non-empty) Variable Inclusion ($\mathrm{RVI}_{\exists}$) if and only if, for some non-empty $\Gamma' \subseteq \Gamma$ implying Δ, $Var^{+}(\Gamma') \subseteq Var^{+}(\Delta)$ and $Var^{-}(\Gamma') \subseteq Var^{-}(\Delta)$

The differences among these various constraints in terms of which arguments are counted as complying with the corresponding version of topic inclusion will be clearer in the next chapter, where several examples will be considered and compared. For the moment, the focus will mainly be on the existential version of variable inclusion as it is the main feature characterizing the central containment logics in the literature.

2.2 Technical Preliminaries

Before characterizing certain systems as variable inclusion logics, some technical preliminaries are needed. The notions introduced in this section will be used throughout the work and will also serve to define more precisely several terms that have already appeared in the preceding informal discussion. Any unexplained notation concerning Universal Algebra and Abstract Algebraic Logic can be found in Burris and Sankappanavar, 1981 and Font, 2016, respectively.

Given a similarity type $\mathscr{L}$ and a countably infinite set Var of generators, the absolutely free algebra $\mathsf{Fml}(\mathscr{L})$ over Var is the formula algebra of type $\mathscr{L}$, whose universe is $Fml(\mathscr{L})$. The elements of Var are the propositional variables or atoms and are denoted with the lowercase Latin letters $p, q, r, \ldots$. The elements of $Fml(\mathscr{L})$ are the $\mathscr{L}$-formulas, or simply formulas, which we denote with lowercase Greek letters $\varphi, \psi, \pi, \ldots$, while sets of formulas are denoted with uppercase Greek letters $\Gamma, \Pi, \Sigma, \ldots$.

Definition 2.2.1. A *logic* $\mathbf{L}$ of type $\mathscr{L}$ is a pair $\mathbf{L} = \langle \mathsf{Fml}(\mathscr{L}), \vdash_{\mathbf{L}} \rangle$. $\mathsf{Fml}(\mathscr{L})$ is a formula algebra and $\vdash_{\mathbf{L}} \subseteq \mathscr{P}(Fml(\mathscr{L})) \times \mathscr{P}(Fml(\mathscr{L}))$ is a substitution-invariant multiple conclusion consequence relation[5].

Moreover, a logic $\mathbf{L}$ is *finitary* if and only if $\Gamma \vdash_{\mathbf{L}} \Delta$ entails that for some finite Γ', Δ' such that $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, $\Gamma' \vdash_{\mathbf{L}} \Delta'$.

Although most of the discussion in this work employs a multiple premises-multiple conclusions framework, two alternative inferential settings need to be introduced as well. The first is the single premise-single conclusion framework, which only counts inferences where premises and conclusions are either singletons or empty. The second is the so-called formula-formula framework, which further restricts this last setting by considering only singletons on both the premise and conclusion side—namely, neither can be empty.

Definition 2.2.2 (Scott, 1974)**.** A *Scott consequence relation* for $\mathscr{L}$ is a (multiple conclusion) consequence relation $\vdash \subseteq \mathscr{P}(Fml(\mathscr{L})) \times \mathrm{P}(Fml(\mathscr{L}))$ that satisfies the following conditions for every $\Gamma, \Delta \subseteq Fml(\mathscr{L})$ and all

[5]Notice that, for the moment, the only restriction for a subset of $\mathscr{P}(Fml(\mathscr{L})) \times \mathscr{P}(Fml(\mathscr{L}))$ to count as a consequence relation is for it to be closed under substitution invariance. Consequence relations are then said to be structural if they also satisfy the conditions in Definition 2.2.2.

$\varphi \in Fml(\mathscr{L})$:

$$\begin{array}{rl}
\text{[Reflexivity]} & \varphi \vdash \varphi \\
\text{[Monotonicity]} & \text{If } \Gamma' \subseteq \Gamma,\ \Delta' \subseteq \Delta \text{ and } \Gamma' \vdash \Delta', \text{ then } \Gamma \vdash \Delta \\
\text{[Transitivity]} & \text{If } \Gamma \vdash \varphi, \Delta \text{ and } \Gamma, \varphi \vdash \Delta, \text{ then } \Gamma \vdash \Delta
\end{array}$$

A logic $\mathbf{L}$ is *structural* only if $\vdash_{\mathbf{L}}$ is a Scott consequence relation[6] A logic $\mathbf{L}$ is *Tarskian* exactly when it is structural with a restriction to the multiple premises-single conclusion framework, namely $\vdash_{\mathbf{L}} \subseteq \mathscr{P}(Fml(\mathscr{L})) \times Fml(\mathscr{L})$.

A formula π is a *theorem* of a logic $\mathbf{L}$ if and only if $\varnothing \vdash_{\mathbf{L}} \pi$. Analogously, a formula π is an *antitheorem* of a logic $\mathbf{L}$ if and only if $\pi \vdash_{\mathbf{L}} \varnothing$. These definitions can be extended to sets, in which case an (*anti*)*theorematic set* Δ (Γ) of a logic $\mathbf{L}$ is such that $\varnothing \vdash_{\mathbf{L}} \Delta$ ($\Gamma \vdash_{\mathbf{L}} \varnothing$). Moreover, notice that if $\mathbf{L}$ is monotonic, then $\Gamma \vdash_{\mathbf{L}}$ ($\vdash_{\mathbf{L}} \Delta$) entails $\Gamma \vdash_{\mathbf{L}} \Sigma$ ($\Sigma \vdash_{\mathbf{L}} \Delta$) for any set of formulas Σ. The following definition serves to equate certain conditional theorems with provable formula–formula inferences.

Definition 2.2.3. Let $\mathbf{L}$ be a logic over a language $\mathscr{L}$ with includes a conditional symbol $\rightarrow$. The logic $\mathbf{L}_{\mathtt{fde}}$ with language $\mathscr{L} \setminus \{\rightarrow\}$ is the *first-degree fragment* of $\mathbf{L}$ if and only if for all $\varphi, \psi \in Fml(\mathscr{L})$:

$$\varphi \vdash_{\mathbf{L}_{\mathtt{fde}}} \psi \text{ if and only if } \vdash_{\mathbf{L}} \varphi \rightarrow \psi$$

For any formula π, let $Var(\pi)$ be the set of propositional variables occurring in π. Similarly, for any set of formulas Π, $Var(\Pi) = \bigcup\{Var(\pi) : \pi \in \Pi\}$. The sets of positive and negative variables occurring in a formula

[6]Notice that other necessary conditions for structurality such as Exchange or Contraction are implicitly encoded in Definition 2.2.2 since only *sets* of formulas are considered.

π, denoted by $Var^+(\pi)$ and $Var^-(\pi)$ respectively, are defined recursively according to the following clauses for a language containing $\neg$, $\wedge$, $\vee$:

- $Var^+(\pi) = Var(\pi)$ and $Var^-(\pi) = \varnothing$, for π atomic.
- $Var^+(\neg\pi) = Var^-(\pi)$ and $Var^-(\neg\pi) = Var^+(\pi)$.
- $Var^+(\pi \star \sigma) = Var^+(\pi) \cup Var^+(\sigma)$ and $Var^-(\pi \star \sigma) = Var^-(\pi) \cup Var^-(\sigma)$, for $\star \in \{\wedge, \vee\}$.

Similarly, for any set of formulas Π, $Var^+(\Pi) = \bigcup\{Var^+(\pi) : \pi \in \Pi\}$ and $Var^-(\Pi) = \bigcup\{Var^-(\pi) : \pi \in \Pi\}$.

Next, for a semantic study of logics matrices are to be defined:

Definition 2.2.4. A *logical matrix* $\mathfrak{M}$ for $\mathscr{L}$ is a pair $\mathfrak{M} = \langle \mathsf{A}, D \rangle$, where A is an algebra of type $\mathscr{L}$ and D is a non-empty subset of the universe A of A.

Definition 2.2.5. A *valuation based on a matrix* $\mathfrak{M} = \langle \mathsf{A}, D \rangle$ for $\mathscr{L}$ is a homomorphism from the formula algebra $\mathsf{Fml}(\mathscr{L})$ of $\mathscr{L}$ onto A. For this reason, the set of valuations based on a matrix $\mathfrak{M}$ is called $\mathrm{Hom}_{\mathfrak{M}}$. One may also just consider the set of valuations over an algebra A when no matrix is specified as $\mathrm{Hom}_{\mathsf{A}}$.

Any logical matrix $\mathfrak{M}$ induces a (multiple conclusions) structural consequence relation $\vDash_{\mathfrak{M}}$—hence a notion of *validity*—in the following manner:

$$\Gamma \vDash_{\mathfrak{M}} \Delta \text{ if and only if for all } v \in \mathrm{Hom}_{\mathfrak{M}},$$
$$\text{if } v(\gamma) \in D \text{ for all } \gamma \in \Gamma, \text{ then } v(\delta) \in D \text{ for some } \delta \in \Delta$$

A class of matrices $\boldsymbol{\mathfrak{M}}$ induces a structural consequence relation $\vDash_{\boldsymbol{\mathfrak{M}}}$ by defining $\Gamma \vDash_{\boldsymbol{\mathfrak{M}}} \Delta$ if and only if $\Gamma \vDash_{\mathfrak{M}} \Delta$ for all $\mathfrak{M} \in \boldsymbol{\mathfrak{M}}$.

Given a class of matrices $\mathfrak{M}$, a formula π is a *tautology* in $\mathfrak{M}$ if an only if it is designated in all valuations $v \in \mathrm{Hom}_{\mathfrak{M}}$, *i.e.* $\varnothing \vDash_{\mathfrak{M}} \pi$. Similarly, a formula π is an *antitautology* in $\mathfrak{M}$ if and only if it does not receive a designated value in any valuation $v \in \mathrm{Hom}_{\mathfrak{M}}$, *i.e.* $\pi \vDash_{\mathfrak{M}} \varnothing$. Just as in the case of theorems and antitheorems, these definitions can be extended to sets: a *(anti)tautologous set* Δ (Γ) in $\mathfrak{M}$ is such that $\varnothing \vDash_{\mathfrak{M}} \Delta$ ($\Gamma \vDash_{\mathfrak{M}} \varnothing$).

There are some important results concerning the relation between matrices and logics. The first was proved by Wójcicki, 1988 (Theorem 3.1.5 and Corollary 3.1.6) for single-conclusion inferences, then extended to multiple-conclusions in *e.g.* Chemla and Égré, 2019 and Da Ré and Fiore, n.d.

Theorem 2.2.1 (Generalized Wójcicki)**.** *For every structural logic* $\mathbf{L}$ *there is a (possibly infinite) class of logical matrices* $\mathfrak{M}$ *such that* $\Gamma \vdash_{\mathbf{L}} \Delta$ *if and only if* $\Gamma \vDash_{\mathfrak{M}} \Delta$.

When such correspondence holds for some logic $\mathbf{L}$ and some class of matrices $\mathfrak{M}$, namely that $\Gamma \vdash_{\mathbf{L}} \Delta$ if and only if $\Gamma \vDash_{\mathfrak{M}} \Delta$, $\mathbf{L}$ is said to be characterized or induced by $\mathfrak{M}$ and, in some contexts, $\mathfrak{M}$ may just be replaced by $\mathbf{L}$ (for instance, writing $\Gamma \vDash_{\mathbf{L}} \Delta$ instead of $\Gamma \vDash_{\mathfrak{M}} \Delta$).

For the last result connecting logics and matrices, the notion of *cancellation property* has to be introduced.

Definition 2.2.6. A logic $\mathbf{L} = \langle \mathsf{Fml}(\mathscr{L}), \vdash_{\mathbf{L}} \rangle$ has the *cancellation property* if and only if, for all $\Gamma_i, \Delta_i \subseteq Fml(\mathscr{L}), i \in I$ such that $Var(\Gamma_i \cup \Delta_i) \bigcap Var(\Gamma_j \cup \Delta_j) = \varnothing$ for each $i \neq j$:

$$\text{If} \quad \bigcup_{i \in I} \Gamma_i \vdash_{\mathbf{L}} \bigcup_{i \in I} \Delta_i \quad \text{then} \quad \Gamma_i \vdash_{\mathbf{L}} \Delta_i \ \text{ for some } i \in I$$

Notice that this definition, due to Shoesmith and Smiley, 1978, generalizes the same notion for single premises as it can be found in Wójcicki, 1988. This can be easily seen setting some Δ_i to be a singleton and for each $j \neq i$ setting $\Delta_j = \varnothing$. This property is important for the following characterization (Theorem 15.2 in Shoesmith and Smiley, 1978):

Theorem 2.2.2 (Cancellation Property Theorem)**.** *A finitary structural logic* **L** *is characterized by a unique matrix if and only if* **L** *has the cancellation property*[7]*.*

This result is useful because, for logics lacking the cancellation property––as in the case of some systems discussed in this work––it is impossible to characterize them by means of a single matrix. Nevertheless, generalized Wójcicki's theorem ensures that if the logic is structural, it can still be characterized by a suitable class of matrices.

Before moving on to the introduction of the first containment logics in this work, some structures that characterize well-known logics need to be introduced. First, a De Morgan Lattice is an algebra $\mathsf{DML} = \langle DML, \wedge, \vee, \neg \rangle$ of type $\langle 2, 2, 1 \rangle$ such that $\langle DML, \wedge, \vee \rangle$ is a distributive lattice satisfying the identities $\neg\neg x \approx x$, $\neg(x \wedge y) \approx \neg x \vee \neg y$ and $\neg(x \vee y) \approx \neg x \wedge \neg y$. De Morgan lattices form a variety which is generated by the 4-element De Morgan lattice DM_4 defined over the set $\{t, b, n, f\}$ and depicted in Figure 2.1. The only proper non-trivial subalgebras (up to isomorphism) of DM_4 are the 2-element Boolean algebra B_2 and 3-element Kleene algebra K_3 (Kalman, 1958). The first is isomorphic to the subalgebra of DM_4 with universe $\{t, f\}$, while the second, is isomorphic to the subalgebra of DM_4 with

[7]Shoesmith and Smiley, 1978 also prove that this characterization can be extended to any non-finitary structural logic **L** which is *stable*, namely, such that the closure of consequence in **L** under substitution and Monotonicity (see Definition 2.2.2) is always a structural consequence relation.

universe $\{t, f, n\}$ or, equivalently, $\{t, f, b\}$. The matrices $\mathcal{M}^{\mathbf{CL}} = \langle \mathsf{B}_2, \{t\} \rangle$, $\mathcal{M}^{\mathbf{K_3}} = \langle \mathsf{K}_3, \{t\} \rangle$, $\mathcal{M}^{\mathbf{LP}} = \langle \mathsf{K}_3, \{t, b\} \rangle$, and $\mathcal{M}^{\mathbf{BD}} = \langle \mathsf{DM}_4, \{t, b\} \rangle$ characterize, respectively, Classical Logic **CL**, Strong Kleene Logic $\mathbf{K_3}$ (Kleene, 1952), the Logic of Paradox **LP** (Asenjo, 1953, Priest, 1979) and Belnap-Dunn's Logic **BD** (Dunn, 1976, Belnap, 1977b, Belnap, 1977a).

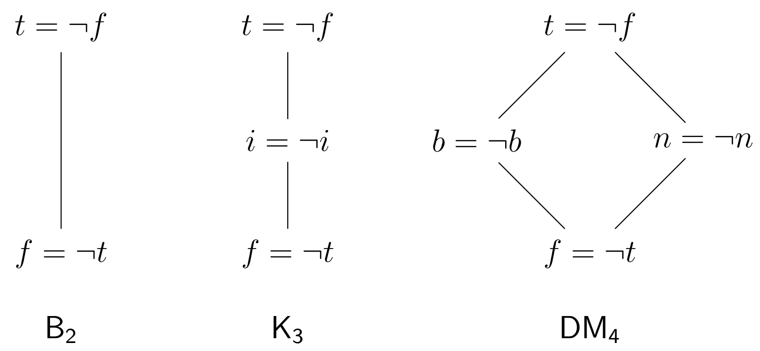

Figure 2.1 The 2-, 3- and 4-element algebras B_2, K_3, and DM_4

Finally, the last structures that need to be introduced are the following. First, a join(meet)-semilattice is an algebra of type $\langle 2 \rangle$ with the binary operation $\vee(\wedge)$ which is commutative, idempotent and associative. Also semilattices form a variety which is generated by the two-element semilattice S_2. Second, a bisemilattice (first introduced in Płonka, 1967b as a generalization of lattices and semilattices) is an algebra of type $\langle 2, 2 \rangle$ such that its $\wedge-$reduct and $\vee-$reduct are semilattices and it is distributive if $\wedge$ and $\vee$ distribute over each other. Notably, in bisemilattices, unlike lattices, absorption may fail. Third, an involutive semilattice is just a semilattice extended with an involutive negation which distributes over the binary operation (namely, a negation satisfying $\neg\neg x \approx x$ and $\neg(x \vee y) \approx \neg x \vee \neg y$). Also this latter kind of algebras form a variety which has three non-trivial subdirectly irreducible members (as it is proved in Crvenković and Dolinka, 2002): IS_2, IS_3, IS_4, displayed in Figure 2.2. If a negation is added to distributive *bi*semilattices such that it is involutive, satisfies De Morgan equalities and $x \wedge (\neg x \vee y) \approx x \wedge y$, then one obtains involutive bisemilattices, further discussed in the next section.

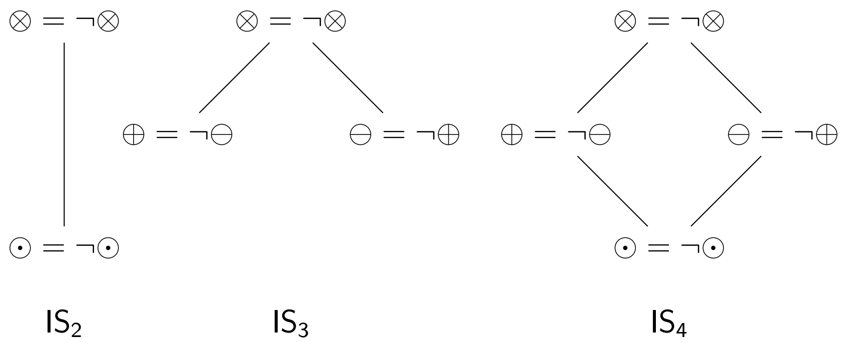

Figure 2.2 The 2-, 3 and 4- element involutive semilattices

2.3 Weak Kleene Logics and their infectious siblings

Weak Kleene logics (Kleene, 1952) are the first containment logics to be presented here. They play a central role in the literature on variable inclusion logics and serve to introduce several notions that will be fundamental throughout this work. Furthermore, they provide a natural starting point for the study of containment logics, given their connection with **CL**, which will be discussed in this section.

There are two Weak Kleene logics: *paracomplete Weak Kleene* $\mathbf{K_3^w}$ and *paraconsistent Weak Kleene* **PWK**. WK logics can be presented proof-theoretically through non-standard sequent calculi (in the sense defined in Bonzio and Pra Baldi, 2017)[8], hence those that can be found in the

[8]They can also be presented as the *external* consequence relations (Avron, 1988, 1991) of *standard* sequent calculi, as done in Paoli and Pra Baldi, 2020b for **PWK** and in Da Ré et al., 2024 for $\mathbf{K_3^w}$.

literature are n-sided (Bonzio and Pra Baldi, 2017, Fjellstad, 2020) or make use of linguistic restrictions (M. Coniglio and Corbalán, 2013), or, as in Borzi and Zirattu, 2026b, include impure rules. A discussion of the technical and conceptual differences among these options can be found in Borzi and Zirattu, 2026b.

Semantically, these systems can both be characterized by matrices with the same algebraic reduct, the 3-element Weak Kleene Algebra WK, which can be defined as follows:

Definition 2.3.1. The *Weak Kleene algebra* is the algebra WK of type $\langle 2, 2, 1\rangle$ with universe $\{t, e, f\}$ and whose operations are depicted in Table 2.1 (Kleene, 1952, p. 334).

	$\neg_{\mathsf{WK}}$
t	f
e	e
f	t

$\wedge_{\mathsf{WK}}$	t	e	f
t	t	e	f
e	e	e	e
f	f	e	f

$\vee_{\mathsf{WK}}$	t	e	f
t	t	e	t
e	e	e	e
f	t	e	f

Table 2.1: The Weak Kleene operations

The matrices $\mathcal{M}^{\mathbf{K_3^w}} = \langle \mathsf{WK}, \{t\}\rangle$ and $\mathcal{M}^{\mathbf{PWK}} = \langle \mathsf{WK}, \{t, e\}\rangle$ induce the WK logics $\mathbf{K_3^w}$ and $\mathbf{PWK}$, respectively:

Definition 2.3.2 ($\mathbf{K_3^w}$-validity)**.** $\Gamma \vDash_{\mathbf{K_3^w}} \Delta$ iff for every valuation $v \in \mathrm{Hom}_{\mathsf{WK}}$: if $v(\gamma) = t$ for all $\gamma \in \Gamma$, then $v(\delta) = t$ for some $\delta \in \Delta$.

Definition 2.3.3 (**PWK**-validity)**.** $\Gamma \vDash_{\mathbf{PWK}} \Delta$ iff for every $v \in \mathrm{Hom}_{\mathsf{WK}}$: if $v(\gamma) \neq f$ for all $\gamma \in \Gamma$, then $v(\delta) \neq f$ for some $\delta \in \Delta$.

WK logics belong to a family of many-valued logics known as *infectious* or *contaminant logics*, which are central within systems of variable inclusion. As the name indicates, the distinctive feature of this family of systems is that at least some of their semantic presentations include a value that propagates in an 'infectious' way. Formally (see, *e.g.* Lakser et al., 1972, Szmuc, 2017):

Definition 2.3.4. An algebra A with universe A has an *infectious element* e (alternatively called an *absorbing* or *zero* element) if and only if for every n-ary operation $\star$ of A, and every $\{x_1, ..., x_n\} \subseteq A$: if $e \in \{x_1, ..., x_n\}$, then $\star^{\mathsf{A}}(x_1, ..., x_n) = e$.

The contaminant behavior of such value is reflected in the following principle:

Fact 2.3.1 (Contamination Principle)**.** Given an algebra A of type $\mathscr{L}$ with an infectious element e, for every formula $\varphi \in \mathsf{Fml}(\mathscr{L})$ and every valuation $v \in \mathrm{Hom}_{\mathsf{A}}$:

$$v(\varphi) = e \text{ if and only if there is } p \in Var(\varphi) \text{ such that } v(p) = e$$

Proof. See proof of Fact 2.1 in Ciuni and Carrara, 2019. □

Moreover, given an algebra A, it can be extended with an infectious element in the following manner:

Definition 2.3.5. Given an algebra A and an element $i \notin A$, its *infectious extension with* i, denoted $\mathsf{A}[i]$, is the algebra with universe $A \cup \{i\}$ whose operations are defined by:

$$\star^{\mathsf{A}[i]}(a_1, ..., a_n) = \begin{cases} i & \text{if } i \in \{a_1, ..., a_n\} \\ \star^{\mathsf{A}}(a_1, ..., a_n) & \text{otherwise} \end{cases}$$

With this definition at hand, it is easy to check that WK can be obtained as the infectious extension of B_2 with e.

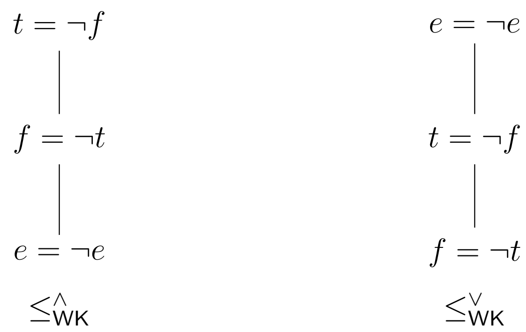

Figure 2.3 The WK algebra as a generalized involutive bisemilattice

The WK algebra is a notable example of an involutive bisemilattice[9], as it generates this variety (Kalman, 1971). Given what was presented at the end of the previous section, a relevant feature of WK is that absorption fails and thus $\wedge_{\mathsf{WK}}$ and $\vee_{\mathsf{WK}}$ do not form a lattice.

In Figure 2.3 The WK-conjunction $\wedge_{\mathsf{WK}}$ coincides with minimum of the order $\leq^{\wedge}_{\mathsf{WK}}$, whilst the WK-disjunction WK corresponds to the the maximum of $\leq^{\vee}_{\mathsf{WK}}$, *i.e.*

$$x \leq^{\wedge}_{\mathsf{WK}} y \text{ iff } x \wedge_{\mathsf{WK}} y = x \text{ iff } \mathtt{min}^{\wedge}(x, y) = x$$

$$y \leq^{\vee}_{\mathsf{WK}} x \text{ iff } x \vee_{\mathsf{WK}} y = x \text{ iff } \mathtt{max}^{\vee}(x, y) = x.$$

More generally, the orders induced by meet and join of algebra with a bisemilattice reduct (such as FA defined below, see Figure 2.4) can be defined in the same way, replacing $\wedge_{WK}$ and $\vee_{WK}$ with the meet and join taken into consideration.

[9]To be more precise, given that its language does not include constants—1 and 0—it is what Paoli and Pra Baldi, 2020a call a *generalized involutive bisemilattice*.

	$\neg_{\mathsf{S}}$
t	f
e	e
f	t

$\wedge_{\mathsf{S}}$	t	e	f
t	t	t	f
e	t	e	f
f	f	f	f

$\vee_{\mathsf{S}}$	t	e	f
t	t	t	t
e	t	e	f
f	t	f	f

Table 2.2: The Sobociński operations

The orders $\leq^{\wedge}_{\mathsf{WK}}$ and $\leq^{\vee}_{\mathsf{WK}}$ coincide, respectively, with the orders $\leq^{\vee}_{\mathsf{S}}$ and $\leq^{\wedge}_{\mathsf{S}}$ induced by the so-called Sobociński disjunction $\vee_{\mathsf{S}}$ and conjunction $\wedge_{\mathsf{S}}$ (see Table 2.2) over the three-element Sobociński algebra S introduced by Sobociński, 1952, which is also an involutive bisemilattice (see Da Ré and Szmuc, 2021). The relation between the operators over WK and S is summarized by this Fact:

Fact 2.3.2.

$$x \leq^{\vee}_{\mathsf{WK}} y \text{ iff } x \wedge_{\mathsf{S}} y = x \text{ iff } \mathtt{min}^{\vee}(x, y) = x$$

$$y \leq^{\wedge}_{\mathsf{WK}} x \text{ iff } x \vee_{\mathsf{S}} y = x \text{ iff } \mathtt{max}^{\wedge}(x, y) = x$$

The introduction of WK logics also serves to present some related systems that will be central in the following chapters and that have been widely discussed in the literature, not only on containment logics. First, just as the underlying algebra of $\mathbf{K^w_3}$ and **PWK**, WK, can be seen as the infectious extension of B_2, there are other logics that can be semantically characterized employing structures obtained in a similar way. For instance, the infectious extension of the 3-element Kleene Algebra K_3, call it $\mathsf{K}_3[i]$, is the algebraic reduct of three matrices characterizing three different logics: $\langle \mathsf{K}_3[i], \{t\} \rangle$ inducing $\mathbf{S}_{\mathtt{etl}}$ (`etl` standing for *exactly true*, in Belikov and

Petrukhin, 2020[10]), $\langle \mathsf{K}_3[i], \{t, b, i\}\rangle$ inducing $\mathbf{S}_{\mathtt{nfl}}$ (nfl standing for *non-falsity* in Belikov and Petrukhin, 2020), $\langle \mathsf{K}_3[i], \{t, b\}\rangle$ inducing $\mathbf{S}_{\mathtt{fde}}$ (also in Belikov and Petrukhin, 2020 and Ferguson, 2017[11]). If, instead of K_3, the infectious extension is performed on BD thus obtaining $\mathsf{BD}[i]$, then again other matrices can be defined, one of them in particular, *i.e.* $\langle \mathsf{BD}[i], \{t, b\}\rangle$, inducing the first degree entailment of Daniel's logic from Daniels, 1990, called $\mathbf{S}^{\star}$ in Ferguson, 2017[12]. Finally, the first-degree fragment of the logic of Analytic Containment **AC** by Angell, 1989 can be captured by the 9-element construction depicted in Figure 2.4, introduced in Ferguson, 2016 and named Ferguson Algebra (hereafter FA) in Paoli et al., 2026, which can be obtained as a *twist product* of the $\neg$-free reduct of WK (see Ferguson, 2016 for details on this construction). FA is the algebraic reduct of the 9-valued semantics for $\mathbf{AC}_{\mathtt{fde}}$, which is proved in Ferguson, 2016 to be induced by the matrix $\langle \mathsf{FA}, \{\langle t, t\rangle, \langle t, f\rangle, \langle t, e\rangle\}\rangle$.

For the purposes of this work, the most important feature of WK logics and their infectious siblings is their characterization with respect to containment constraints[13]. In order to present these characterizations, the notion of 'companion' of a given logic with respect to some variable inclusion requirement needs to be introduced.

[10] $\mathbf{S}_{\mathtt{etl}}$ is also named $\mathbf{L}_{\mathtt{sw}}$ in Martí and Martínez-Fernández, 2021 for the strong-weak behavior of its non-classical values

[11] It is shown in Ferguson, 2017, p.30 that $\mathbf{S}_{\mathtt{fde}}$ coincides with the first-degree fragment of Deutsch's logic of paraconsistent analytic implication S from Deutsch, 1984, independently presented also by Oller, 1999

[12] $\mathbf{S}^{*}_{\mathtt{fde}}$ was also independently introduced by Graham Priest in Priest, 2010 as $\mathbf{FDE}_{\varphi}$

[13] As done in *e.g.* Ferguson, 2015, Bonzio and Pra Baldi, 2021 and Bonzio et al., 2022, Ciuni and Carrara, 2019, Ciuni and Carrara, 2016

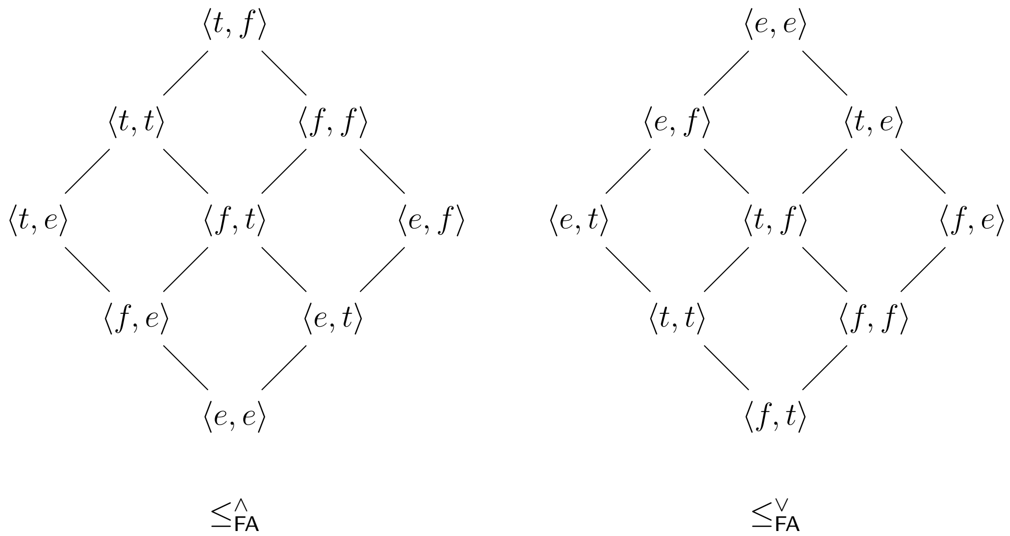

Figure 2.4 Ferguson Algebra

2.4 Weak (Balanced) Variable Inclusion Companions

As discussed in section 1.2, once a formal account of content inclusion is established, the next question concerns which logical systems qualify as content inclusion logics. Since content inclusion has been characterized in terms of both truth preservation and subject-matter preservation—where the latter can be formalized according to the preferred specification in the list of section 2.1, while preservation of truth can be intended as validity in some logical framework—one way to approach this question is by identifying logics that serve as *companions* to other systems. This means showing that the target logic validates inferences that, on the one hand, preserve certain alethic values according to the standard of a given logic—–of which it is considered a companion—–and, on the other hand, satisfy a form of topic preservation expressed through variable inclusion. Depending on the kind of topic inclusion that is satisfied by a logic, such logic coincides with a corresponding kind of companion with respect to a system. Moreover, this characterization allows to include such logic in the family of variable inclusion logics or, under the previously discussed interpretation, content inclusion logics.

Various kinds of variable inclusion companions will be discussed throughout this and the next chapters, starting from the systems which can be seen as complying with $\mathbf{RVI}_\exists$ or $\mathbf{LVI}_\exists$—or their balanced versions $\mathbf{RVI}^{\pm}_\exists$ or $\mathbf{LVI}^{\pm}_\exists$—but, importantly, admitting some exceptions. In some works (*e.g.* Bonzio and Pra Baldi, 2021, Bonzio et al., 2022, Paoli et al., 2021, Szmuc and Rubin, 2022) these logics simply received the name of right or left variable inclusion companions. However, in order to distinguish them from other companions that will be introduced later, they will be called here *Weak Variable Inclusion Logics* for the regular case, *Weak Balanced Variable Inclusion Logics* for the balanced case.

Definition 2.4.1. Given a logic $\mathbf{L}$, its *weak right companion* is the logic $\mathbf{L}^{\mathbf{wr}}$ characterized as follows:

$$\Gamma \vdash_{\mathbf{L}^{\mathbf{wr}}} \Delta \text{ iff } \Gamma \vdash_{\mathbf{L}} \Delta \text{ and } \begin{cases} \text{(i) } \Gamma \vdash_{\mathbf{L}} \varnothing \text{ or} \\ \text{(ii) } \exists \Delta' \mid \Delta' \subseteq \Delta \text{ and } \Delta' \neq \varnothing, \text{ s.t.} \\ \qquad \Gamma \vdash_{\mathbf{L}} \Delta' \text{ and } Var(\Delta') \subseteq Var(\Gamma) \end{cases}$$

Definition 2.4.2. Given a logic $\mathbf{L}$, its *weak left companion* is the logic $\mathbf{L}^{\mathbf{wl}}$

characterized as follows:

$$\Gamma \vdash_{\mathbf{L^{wl}}} \Delta \text{ iff } \Gamma \vdash_{\mathbf{L}} \Delta \text{ and } \begin{cases} \text{(i) } \varnothing \vdash_{\mathbf{L}} \Delta \text{ or} \\ \text{(ii) } \exists \Gamma' \mid \Gamma' \subseteq \Gamma \text{ and } \Gamma' \neq \varnothing, \text{ s.t.} \\ \quad \Gamma' \vdash_{\mathbf{L}} \Delta \text{ and } Var(\Gamma') \subseteq Var(\Delta) \end{cases}$$

Some remarks are in order. Both definitions show that the Weak Companion of a logic coincides with a restriction of its inferences to those respecting one of the two clauses. Clause (ii) corresponds to a version of the existential (non-empty) variable inclusion requirement—$\mathbf{RVI}_{\exists}$ or $\mathbf{LVI}_{\exists}$, respectively—while clause (i) describes the exceptions to the same. In the first case, the clause expresses the fact that if the original logic has antitheorems, so will its companion. In the second case, it expresses that if the original logic has theorems, they are also theorems of its companion. While these in principle do not constitute violations of the variable inclusion requirement, they do in the context of structural consequence relations. In fact if, given some structural logic $\mathbf{L}$, $\Gamma \vdash_{\mathbf{L}} \varnothing$, then $\Gamma \vdash_{\mathbf{L}} \Delta$ for any Δ, hence $\Gamma \vdash_{\mathbf{L^{wr}}} \Delta$ for any Δ, even when $Var(\Delta) \nsubseteq Var(\Gamma)$—and analogously for theorems. Similar remarks apply to the weak balanced companions:

Definition 2.4.3. Given a logic $\mathbf{L}$, its *weak balanced right companion* is the logic $\mathbf{L^{wr\pm}}$ characterized as follows:

$$\Gamma \vdash_{\mathbf{L^{wr\pm}}} \Delta \text{ iff } \Gamma \vdash_{\mathbf{L}} \Delta \text{ and } \begin{cases} \text{(i) } \Gamma \vdash_{\mathbf{L}} \varnothing \text{ or} \\ \text{(ii) } \exists \Delta' \mid \Delta' \subseteq \Delta \text{ and } \Delta' \neq \varnothing \text{ s.t. } \Gamma \vdash_{\mathbf{L}} \Delta' \\ \text{ and } Var^{+}(\Delta') \subseteq Var^{+}(\Gamma),\ Var^{-}(\Delta') \subseteq Var^{-}(\Gamma) \end{cases}$$

Definition 2.4.4. Given a logic $\mathbf{L}$, its *weak balanced left companion* is the logic $\mathbf{L}^{\mathbf{wl}\pm}$ characterized as follows:

$$\Gamma \vdash_{\mathbf{L}^{\mathbf{wl}\pm}} \Delta \text{ iff } \Gamma \vdash_{\mathbf{L}} \Delta \text{ and } \begin{cases} \text{(i) } \varnothing \vdash_{\mathbf{L}} \Delta \text{ or} \\ \text{(ii) } \exists \Gamma' \mid \Gamma' \subseteq \Gamma \text{ and } \Gamma' \neq \varnothing \text{ s.t. } \Gamma' \vdash_{\mathbf{L}} \Delta \\ \text{ and } Var^{+}(\Gamma') \subseteq Var^{+}(\Delta),\ Var^{-}(\Gamma') \subseteq Var^{-}(\Delta) \end{cases}$$

There is a general construction that can be employed in order to obtain the Weak variable inclusion companion of a structural logic **L**. Infectious extensions are used both for the regular and balanced systems, though for the latter some extensions are needed. First, the regular case will be addressed, postponing the balanced to the end of the section.

Definition 2.4.5. Given any logical matrix $\mathfrak{M} = \langle \mathsf{A}, D \rangle$, and any element $e \notin A$—A being the universe of A—the *l-infectious extension* of $\mathfrak{M}$ with e is the matrix $\mathfrak{M}^l[e] = \langle \mathbf{A}[e], D \cup \{e\} \rangle$, while its r-infectious extension is the matrix $\mathfrak{M}^r[e] = \langle \mathbf{A}[e], D \rangle$.

These definitions extend to a class of matrices $\boldsymbol{\mathfrak{M}}$, whose l-infectious extension is $\boldsymbol{\mathfrak{M}}^l[e]$ obtained replacing each $\mathfrak{M}_i \in \boldsymbol{\mathfrak{M}}$ by $\mathfrak{M}_i^l[e]$ and whose r-infectious extension is $\boldsymbol{\mathfrak{M}}^r[e]$ obtained replacing each $\mathfrak{M}_i \in \boldsymbol{\mathfrak{M}}$ by $\mathfrak{M}_i^r[e]$.

Given these definitions, the following results can be stated—proved *e.g.* in Ciuni et al., 2018 (Theorems 1 and 2) and in Bonzio et al., 2022 (Theorems 6.1.8. and 5.1.9)[14] :

[14] Bonzio et al., 2022 use a construction, of which the one employed here is a special example, obtained using the so-called Płonka sums of algebras, a tool first introduced by Płonka, 1967a widely used in the literature of variable inclusion logics. Moreover, the framework they consider is only for single conclusion inferences, while Ciuni et al., 2018 only consider logics characterized by a single matrix, but their proofs are easily extendable to the more general case.

Theorem 2.4.1.

$$\Gamma \vDash_{\mathfrak{M}^{l}[e]} \Delta \textit{ iff } \Gamma \vDash_{\mathfrak{M}} \Delta \textit{ and } \begin{cases} (i)\ \varnothing \vDash_{\mathfrak{M}} \Delta \ \textit{or} \\ (ii)\ \exists \Gamma' \mid \Gamma' \subseteq \Gamma \textit{ and } \Gamma' \neq \varnothing,\ \textit{s.t.} \\ \qquad \Gamma' \vDash_{\mathfrak{M}} \Delta \textit{ and } Var(\Gamma') \subseteq Var(\Delta) \end{cases}$$

Theorem 2.4.2.

$$\Gamma \vDash_{\mathfrak{M}^{r}[e]} \Delta \textit{ iff } \Gamma \vDash_{\mathfrak{M}} \Delta \textit{ and } \begin{cases} (i)\ \Gamma \vDash_{\mathfrak{M}} \varnothing \ \textit{or} \\ (ii)\ \exists \Delta' \mid \Delta' \subseteq \Delta \textit{ and } \Delta' \neq \varnothing,\ \textit{s.t.} \\ \qquad \Gamma \vDash_{\mathfrak{M}} \Delta' \textit{ and } Var(\Delta') \subseteq Var(\Gamma) \end{cases}$$

Then, given that generalized Wójcicki's theorem guarantees that each structural logic can be characterized by a class of matrices, this construction give a recipe to obtain the weak left and weak right companions of any structural logic.

WK logics are an example of this sort and serve as a useful instantiation of the features of weak companions in terms of exceptions to relevance. Urquhart (Theorem 2.3.1. in Urquhart, 2001) first proved that $\mathbf{K_3^w}$ is the weak right companion of **CL** (though for single conclusion inferences), while Ciuni and Carrara (Fact 3.1 and Theorem 3.8 Ciuni and Carrara, 2016) proved the dual result for **PWK**, showing that it is the weak left companion of **CL**. These results are semantic characterizations, namely, they apply to the notion of validity defined over the matrices $\mathfrak{M}^{\mathbf{K_3^w}}$ and $\mathfrak{M}^{\mathbf{PWK}}$ with respect to that defined over $\mathfrak{M}^{\mathbf{CL}}$. However, given that these structures induce the logics $\mathbf{K_3^w}$, **PWK** and **CL** respectively, their characterizations can be equivalently expressed in syntactic terms as in Definitions 2.4.1 and 2.4.2.

Fact 2.4.1 (Characterization of $\mathbf{K_3^w}$).

$$\Gamma \vDash_{\mathbf{K_3^w}} \Delta \text{ iff } \Gamma \vDash_{\mathbf{CL}} \Delta \text{ and } \begin{cases} \text{(i) } \Gamma \vDash_{\mathbf{CL}} \varnothing \text{ or} \\ \text{(ii) } \exists \Delta' \mid \Delta' \subseteq \Delta \text{ and } \Delta' \neq \varnothing \text{, s.t.} \\ \quad \Gamma \vDash_{\mathbf{CL}} \Delta' \text{ and } Var(\Delta') \subseteq Var(\Gamma) \end{cases}$$

Fact 2.4.2 (Characterization of **PWK**).

$$\Gamma \vDash_{\mathbf{PWK}} \Delta \text{ iff } \Gamma \vDash_{\mathbf{CL}} \Delta \text{ and } \begin{cases} \text{(i) } \varnothing \vDash_{\mathbf{CL}} \Delta \text{ or} \\ \text{(ii) } \exists \Gamma' \mid \Gamma' \subseteq \Gamma \text{ and } \Gamma' \neq \varnothing \text{, s.t.} \\ \quad \Gamma' \vDash_{\mathbf{CL}} \Delta \text{ and } Var(\Gamma') \subseteq Var(\Delta) \end{cases}$$

These results provide a crucial insight into WK logics and their relation to relevance. First, they explain why arguments such as Examples IIa. and IIb. fail in $\mathbf{K_3^w}$ and **PWK**, respectively. In fact, the inference corresponding to disjunction introduction, also called Addition, fails in $\mathbf{K_3^w}$—as in any Parry system—while conjunction elimination, also called Simplification, fails in **PWK**—as in any dual-Parry system. However, given that WK logics are structural, the failure of these inferences is not reflected by the behavior of the comma disjoining conclusions or conjoining premises, respectively, namely: $\varphi \nvDash_{\mathbf{K_3^w}} \varphi \vee \psi$ and $\varphi \wedge \psi \nvDash_{\mathbf{PWK}} \varphi$, but $\varphi \vDash_{\mathbf{K_3^w}} \varphi, \psi$ and $\varphi, \psi \vDash_{\mathbf{PWK}} \varphi$. In fact, as long as variable inclusion is only required for *some* subset of the conclusions or premises, as it is specified by clause (ii), any other formula may be added to either side which need not respect the containment constraint.

Second, given that **CL** has both theorems and antitheorems, the paracomplete $\mathbf{K_3^w}$ shares the same antitheorems as **CL** while the paraconsistent **PWK** shares classical theorems. Importantly, given that both **CL** and WK logics are structural—in particular, they are monotonic—$\mathbf{K_3^w}$-validity and **PWK**-validity allow for exceptions to variable inclusion each time an antitautology or a tautology, respectively, appears as premise or conclusion. As mentioned in section 2.1 and as will be further discussed in the next chapter, this feature makes problematic the placement of $\mathbf{K_3^w}$ and **PWK**, as well as weak variable inclusion logics more in general, into the family of relevance logics.

Just as infectious extensions allow to obtain the weak companion of a given structural logic, a modification of this construction can be used to obtain its weak *balanced* right of left companion. There are several options in this respect, such as *refined* infectious extensions, introduced in Szmuc and Rubin, 2022[15], as well as *signed* infectious extensions introduced in Randriamahazaka, 2022—though the latter construction can be used to characterize the logics discussed in section 3.1, which may happen to coincide with weak balanced companions but not necessarily so. Thus, the former kind of extension is presented here with a renaming to match the terminology for balanced companions.

In the following, given a set A and an element e such that $e \notin A$, $A[+]$ is defined as $A \cup (A \times \{e\})$ while $A[-]$ stands for $\{e\} \times (A \cup \{e\})$, and, finally, $A[\pm]$ indicates their union. Given an algebra A defined on this universe and assuming that it includes a unary operation $\neg_{\mathsf{A}}$ which is defined for each element (*i.e.* each $x \in A[\pm]$ is such that there is $y = \neg_{\mathsf{A}} x$), two further operators, called π_0 and π_1, may be defined: for $a \in A$, $\pi_0(a) = a$ and $\pi_1(a) = \neg_{\mathsf{A}} a$, for $a = \langle x, y \rangle \in A[\pm] \setminus A$, $\pi_0(a) = x$ and $\pi_1(a) = y$.

[15] Although Szmuc and Rubin, 2022 only show how to obtain the *pure* balanced *right* companion of a logic (see next chapter for their definition), the structures considered here can be easily adapted to the weaker case, as well as to the left-to-right direction of containment.

Additionally, for $a \in A$, let π_2 be the operator such that $\pi_2(a) = b$ where $\neg_{\mathsf{A}} b = a$—and, if there are $b_1 \neq b_2$ such that, for some a, $\neg_{\mathsf{A}} b_1 = \neg_{\mathsf{A}} b_2 = a$, $\pi_2(a)$ chooses indistinctly between them. With this, balanced infectious extensions can be defied as follows:

Definition 2.4.6 (Szmuc and Rubin, 2022)**.** Let A be an algebra of type $\mathscr{L}$ with universe A, and let $\mathsf{A}[e]$ be its infectious extension. Then, its *balanced infectious extension*[16] $\mathsf{A}[\pm]$ is the algebra of type $\mathscr{L}$ with universe $A[\pm]$, defined as follows:

$$\neg_{\mathsf{A}[\pm]} a = \begin{cases} \neg_{\mathsf{A}} a & \text{if } a \in A, \text{ otherwise} \\ \langle \pi_1(a), \pi_0(a) \rangle \end{cases}$$

and for all n-ary operations $\star_{\mathsf{A}[\pm]}$:

$$\star_{\mathsf{A}[\pm]}(a_1, \ldots, a_n) = \begin{cases} \star_{\mathsf{A}}(a_1, \ldots, a_n) \ \text{ if all } a_i \in A, \text{otherwise} \\ \langle \star_{\mathsf{A}[\mathsf{e}]}(\pi_0(a_1), \ldots, \pi_0(a_n)), \ \neg \star_{\mathsf{A}[\mathsf{e}]} (\pi_1(a_1), \ldots, \pi_1(a_n)) \rangle \end{cases}$$

As an example, Figure 2.5 displays the balanced infectious extension of B_2.

Definition 2.4.7. Let $\mathfrak{M} = \langle \mathsf{A}, D \rangle$ be an $\mathscr{L}$-matrix. Its *r-balanced infectious extension* is the matrix $\mathfrak{M}^r[\pm] = \langle \mathsf{A}[\pm], D \cup (D \times \{e\}) \rangle$, while its *l-balanced infectious extension* is the matrix $\mathfrak{M}^l[\pm] = \langle \mathsf{A}[\pm], A[\pm] \setminus \{\overline{D} \cup (\overline{D} \times \{e\})\} \rangle$

These definitions extend to a class of matrices $\boldsymbol{\mathfrak{M}}$, whose r-balanced infectious extension is $\boldsymbol{\mathfrak{M}}^r[\pm]$ obtained replacing each $\mathfrak{M}_i \in \boldsymbol{\mathfrak{M}}$ by $\mathfrak{M}_i^r[\pm]$ and whose l-balanced infectious extension is $\boldsymbol{\mathfrak{M}}^l[\pm]$ obtained replacing each

[16] This construction may also be seen as a De Morgan Płonka sum (Randriamahazaka, 2024) over IS_4 where A is the bottom fiber, the trivial algebra whose only element is $\langle e, e \rangle$ is the top fiber and the two other fibers are two isomorphic copies of A whose indices are one the negation of the other and whose universes are all the elements $\langle a, e \rangle$ (for $a \in A$) and all the elements $\langle e, b \rangle$ (for $b \in A$), respectively.

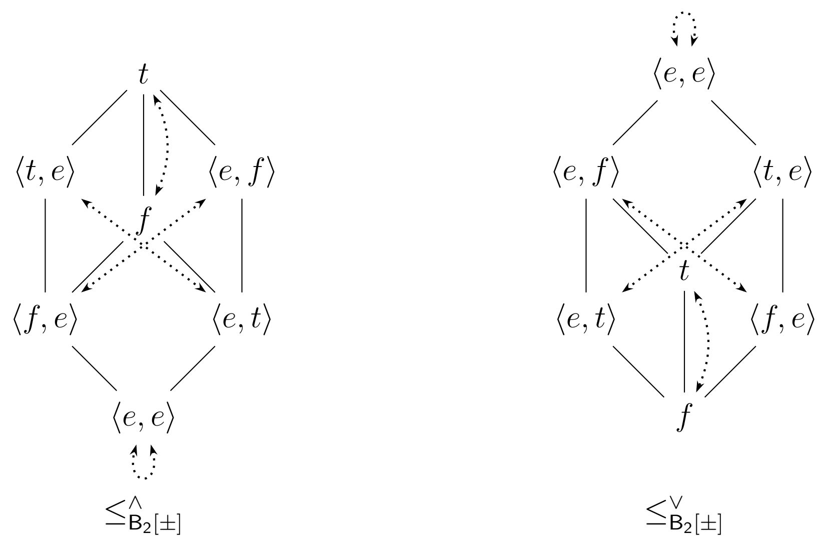

Figure 2.5 $\mathsf{B}_2[\pm]$ (dotted arrows connect each element to its negation).

$\mathfrak{M}_i \in \boldsymbol{\mathfrak{M}}$ by $\mathfrak{M}_i^l[\pm]$.

The notions considered here extend those in Szmuc and Rubin, 2022 both because multiple conclusions are allowed—having important consequences on the characterizability of some systems[17]—and because they allow to prove the result for balanced variable inclusion also for the left-to-right direction, which has been largely disregarded in the literature. The characterization result for weak balanced right companions can be obtained adapting the proof of Theorem 9 in Szmuc and Rubin, 2022 and the same strategy can be employed for the weak balanced left companions. Here only the characterizations are stated, but a more explicit proof is given in chapter 4, Theorem 4.3.1 (transposable to these constructions via the translations in Theorem 4.3.3).

Theorem 2.4.3 (Theorem 9 in Szmuc and Rubin, 2022—adapted version)**.**

[17]This issue will be addressed in the next chapter when *pure* (balanced) companions will be introduced, some of them displaying the cancellation property, only when single conclusion inferences are considered.

$$\Gamma \vDash_{\mathfrak{M}^{r}[\pm]} \Delta \text{ } \textit{iff} \text{ } \Gamma \vDash_{\mathfrak{M}} \Delta \text{ } \textit{and} \begin{cases} \textit{(i)}\ \Gamma \vDash_{\mathfrak{M}} \varnothing \ \textit{or} \\ \textit{(ii)}\ \exists \Delta' \mid \Delta' \subseteq \Delta \ \textit{and}\ \Delta' \neq \varnothing \ \textit{s.t.}\ \Gamma \vDash_{\mathfrak{M}} \Delta' \\ \ \textit{and}\ Var^{+}(\Delta') \subseteq Var^{+}(\Gamma),\ Var^{-}(\Delta') \subseteq Var^{-}(\Gamma) \end{cases}$$

Theorem 2.4.4.

$$\Gamma \vDash_{\mathfrak{M}^{l}[\pm]} \Delta \text{ } \textit{iff} \text{ } \Gamma \vDash_{\mathfrak{M}} \Delta \text{ } \textit{and} \begin{cases} \textit{(i)}\ \varnothing \vDash_{\mathfrak{M}} \Delta \ \textit{or} \\ \textit{(ii)}\ \exists \Gamma' \mid \Gamma' \subseteq \Gamma \ \textit{and}\ \Gamma' \neq \varnothing \ \textit{s.t.}\ \Gamma' \vDash_{\mathfrak{M}} \Delta \\ \ \textit{and}\ Var^{+}(\Gamma') \subseteq Var^{+}(\Delta),\ Var^{-}(\Gamma') \subseteq Var^{-}(\Delta) \end{cases}$$

Just as in the non-balanced case, given generalized Wójcicki's theorem this construction gives a recipe to obtain the balanced weak left and balanced weak right companions of any structural logic.

Before closing the section, it is useful to also display which infectious siblings of WK logics have been shown in the literature[18] to be weak (balanced) variable inclusion companions of some of the logics introduced in section 2.2:

$$\mathbf{K_3^{wr}} = \mathbf{S_{etl}} \quad \mathbf{LP^{wl}} = \mathbf{S_{nfl}} \quad \mathbf{LP^{wr}} = \mathbf{S_{\mathtt{fde}}} \quad \mathbf{BD^{wr}} = \mathbf{S^{*}_{\mathtt{fde}}} \quad \mathbf{BD^{wr\pm}} = \mathbf{AC_{\mathtt{fde}}}$$

2.5 Motivations for infectious Logics

In order to conclude the presentation of WK logics and their infectious siblings, this section introduces some philosophical interpretations that they have received in the literature—which also serve as motivations for the failure of Addition or Simplification— other than the focus on relevance that

[18] The first two were proved, *e.g.*, in Belikov and Petrukhin, 2020. The last three were originally proved in Ferguson, 2015 and Ferguson, 2016. The last was also independently proved in Fine, 2016a.

has been pursued so far. Not only this helps to better grasp the centrality of infectious logics even when other notions of relevance—or none—are preferred, but it also draws more connections with the logics that will be later introduced, given that, as it will be discussed, these can be seen as extending at least some of such motivations.

The first motivation for WK logics is due to Bochvar, 1938 and Halldén, 1949), at the beginning of the past century. These works can be placed within the literature on logics of nonsense or significance logics, namely the attempts to develop systems which can handle grammatical but otherwise meaningless or nonsensical sentences in a formally systematic way. In Bochvar, 1938 and Halldén, 1949, the guiding intuition was that statements describing paradoxical sets are actually meaningless, thus not truth-apt, and consequently neither true nor false. Importantly, not only they considered these statements to be gappy, but also regarded their meaninglessness as an 'infectious' pathology which extends to any compound statement containing these paradoxical sentences. In Bochvar, 1938, logical consequence was defined as preservation of truth, while in Halldén, 1949 it was defined as preservation of non-falsity. Consequently, the fragments of these logics defined over the language $\langle \wedge, \vee, \neg \rangle$—which are the *internal* Bochvar and Halldén's logics[19]—correspond to $\mathbf{K_3^w}$ and **PWK**, respectively. Given this interpretation, the failure of Addition ($\varphi \vDash \varphi \vee \psi$) in $\mathbf{K_3^w}$ can be motivated by the fact that a meaningless conclusion should not follow from true premises (e.g. take φ to be true and ψ to be paradoxical). On the other hand, the failure of Simplification ($\varphi \wedge \psi \vDash \psi$) in **PWK** can be seen as a

[19]The *external* logics of Bochvar and Halldén's include operators which can express what is the semantic value of a formula and where the non-classical value ceased to be infectious.

consequence of the fact that an argument from a nonsensical statement to a false conclusion cannot be valid (consider for instance φ being nonsensical and ψ being false). More generally, the former failure is justified for any logic of nonsense whose semantic presentation includes an infectious value which is not designated, the latter to the case in which it is designated.

Moreover, while Bochvar and Halldén's works focused on paradoxical statements, there are many examples in the literature of other cases of meaninglessness. Among them, the failure of the verifiability criterion formulated in the positivist tradition (*e.g.* in the work of Hempel and Carnap), category mistakes (a term that was first introduced in Ryle, 1949), or even reference failure (*e.g.* in Martí and Martínez-Fernández, 2021 and Prior, 1967). In general, whenever one assumes that (i) only sentences that express a propositional content can be assigned truth and falsity, and (ii) a complex sentence expresses a proposition only if its sentential components express propositions too, nonsensical statements—of whatever kind—will be modeled infectiously and, within a many valued semantics, trough a contaminant value—which, in turn, will provoke one of the above failures depending on the semantic definition of logical consequence.

The meaningless interpretation has also been extended to systems obeying some form of balanced variable inclusion in Szmuc and Rubin, 2022, taking inspiration from the 9-valued semantics for $\mathbf{AC}_{\texttt{fde}}$ and its reading by Ferguson, 2017. The authors argue that there may be meaningless statements, such as "Colorless greens ideas sleep furiously", whose negation, "It is not the case that colorless greens ideas sleep furiously", is nevertheless meaningful. Although this line of thought is not pursued in detail, the underlying moral is that there may be principled reasons to distinguish between a statement and its negation with respect to meaningfulness, since negation can, in principle, be transformative in this respect. Therefore, within a many-valued semantic framework, for a system to be able to handle meaninglessness and transformativeness of negation, the addition of an infectious value is not sufficient. The balanced infectious extension construction by Szmuc and Rubin, 2022 does not only include a range of non-contaminant values—the elements of A—and a contaminant value—$\langle e, e\rangle$—for which negation is transparent, *i.e.* a range of values for meaningful statements with meaningful negations or a value for nonsense statements whose negation is nonsense too. It also includes the mixed cases, namely, for each non-contaminant value there is also the possibility of its negation being contaminant and vice-versa, representing the case in which negation is transformative—that is, all elements of the form $\langle a, e\rangle$ for some $a \in A$, or $\langle e, a\rangle$, respectively. Under this interpretation, as in the case of WK logics, one may choose whether or not to designate nonsense values (whose negations may or may not themselves be nonsense). In the former case, the so-obtained consequence relation may be seen as preserving a set of meaningful values from conclusions to premises—as in weak balanced left companions. In the latter case, this set is preserved in the other direction—as in weak balanced right companions.

A further kind of motivation for both $\mathbf{K_3^w}$ and **PWK** has been given by Fitting, 1994 and Szmuc, 2019, respectively, within an epistemic framework. Both Fitting and Szmuc consider a setting where there is a group of experts expressing their opinion (in the form of yes/no answers) on different issues (statements that can be formalized as formulas). In this framework, there's two possible scenarios: on the one hand, it may happen that experts have indeterminate (neither positive nor negative) opinions, while, on the other hand, experts may entertain inconsistent (both positive and negative) opinions. According to the authors, once all experts conveyed their position regarding the issues under discussion, specific policies should be adopted when pooling their opinions, which apply to one of the two possible scenarios. In both cases, assigning truth to a formula corresponds to the case where all experts say the formula in question is true and no expert says it's false, whilst the assignment of falsity corresponds to the case where all experts say the formula in question is false and none says it's true. For the non-classical value, there are two interpretations available. One addresses the situation in which every expert has expressed a consistent opinion but there is some issue for which they have not expressed any opinion on. The other pertains the dual situation, namely when every expert has expressed a determinate opinion, but may have conveyed an inconsistent opinion towards some statement. In the first scenario, assigning the non-classical value to a formula represents the fact that no expert has expressed an opinion about said formula. Then, according to Fitting, 1994, the chosen policy should 'cut down' on the set of experts taken into account when evaluating formulas, to those that have actually expressed a determinate opinion on the issues at hand. Particularly, in the case of disjunction, one may not want to count the experts as being in favor of $\varphi \vee \psi$, if they have no opinion whatsoever about one of the disjuncts (similarly for conjunction and negation). With this, if consequence relation is defined as truth preservation, then Addition would be invalid: from the fact that all experts think some formula φ is true it does not follow that all experts think $\varphi \vee \psi$ is true, since the group of experts may have no opinion towards ψ. In the second scenario, according to Szmuc, 2019, a 'track down' policy should be adopted, namely one should keep track of the experts who have expressed an inconsistent opinion about the issues under discussion. Under this policy, in the case of conjunction one may want to count experts as being both in favor and against $\varphi \wedge \psi$, if they have expressed an inconsistent opinion about either one of the conjuncts (similarly for the other operators). Again, defining logical consequence as truth preservation entails that Simplification cannot be accepted because, if experts have an inconsistent opinion on a conjunction $\varphi \wedge \psi$, this is compatible with them having a consistent, and possibly negative, opinion on one of the conjuncts, e.g. ψ.

Moving on, there are at least two more kinds of philosophical motivations that have been given for $\mathbf{K_3^w}$ or right variable inclusion logics only. One is the off-topic interpretation by Beall, 2016[20]—and its generalization in Song et al., 2023—the other is the computational interpretation in Ferguson, 2014—further developed in Ciuni et al., 2019.

For the first, only Beall's interpretation will be now introduced while its generalization will be discussed in chapter 4, as it applies to both WK logics under a different semantic presentation than the one considered at the moment. Beall, 2016 assumes the notion of a theory as the set of all claims that logically follow from some given assumptions. Clearly, depending on the underlying logic, some statements will be part of the theory while others won't. So for instance, if the theory in question is Euclidean geometry, taking classical consequence for the base logic implies that not only the

[20]Beall's interpretation has been criticized in Francez, 2019. These criticisms are addressed in Carrara and Pra Baldi, n.d., where a defense of the off-topicality interpretation of the infectious value of $\mathbf{K_3^w}$ is proposed.

statement "two points determine a straight line" is part of the theory, but also "two points determine a straight line or the moon is made of green cheese" (as in the famous example by Parry, 1968, p.151). In general, if a theory contains φ, then it will also include $\varphi \vee \psi$, although ψ might be a statement on a completely off-topic subject with respect to the theory. In order to remain on-topic when building a theory, one possibility is to consider a logic that assigns an off-topic value to each statement that is not about what the chosen theory is about and then treat this off-topical status as infectious—as, according to Beall, it spreads just as meaninglessness. Truth preservation in this framework (which is incarnated in $\mathbf{K_3^w}$), will ensure that the theory contains all and only those statements which follow from the assumptions but additionally do not go off-topic.

Finally, Ferguson, 2014 develops a computational interpretation for $\mathbf{S}_{\mathtt{fde}}$ and $\mathbf{S}^{\star}_{\mathtt{fde}}$. Ferguson's proposal is inspired by Belnap's classical reflections on how a computer ought to reason (Belnap, 1977b). In Belnap's setting, the truth value of a statement is taken to coincide with the information that some computer may retrieve about that sentence being true or false. Importantly, this framework allows for incomplete or inconsistent characterizations of a statement if the computer retrieves either no information on the truth or falsity of that statement, or contradictory information on the same. According to Belnap, this motivates the consideration of the gappy and glutty values which are characteristic of **BD**. This idea is extended by Ferguson's consideration of a "faulty computer", which may *fail* to retrieve the value of a given sentence. According to the author, this case is fundamentally different from the situation in which the computer successfully retrieves information which happens to be incomplete. A failure to obtain the relevant value—perhaps due to some error as memory crash or a hardware malfunction—must therefore be distinguished from a successful retrieval that yields a gappy value. This distinction is marked with the infectious character of the former kind of error: indeed, although a computer may succeed in retrieving the value of some statement φ and even be informed that it is true, it might encounter a critical error or crash while retrieving the value of ψ, making the retrieval of $\varphi \vee \psi$ subject to the same error. This idea is extended in Ciuni et al., 2019 where the authors consider the possibility of different kinds of critical errors interacting when they may be hierarchically ordered basing on their level of infectiousness.

Chapter 3

Refining Variable Inclusion

The previous chapter introduced WK logics and their infectious siblings, as well as their characterization as logics of variable inclusion which, on the one hand, comply with one version of such containment and, on the other, admit exceptions to such syntactic constraints when anti-theorems or theorems are present. In this chapter, these two issues will be addressed separately and, then, in a unified framework. The discussion develops from certain discontents——already partly anticipated in section 2.3——about both points. These reservations not only arise when WK logics, or weak variable inclusion logics in general, are viewed as genuine logics of content inclusion, but also from other interpretations that have been given to such systems, as those in section 2.5. In the following, the systems to be introduced are proper subsystems of WK logics whose characterization constitutes a refinement of the ones for $\mathbf{K_3^w}$ and **PWK** with respect to the satisfaction of the variable inclusion proviso—which does not necessarily involve a balanced version thereof.

Some of the material of this chapter draws on joint work with Agustina Borzi (SADAF–CONICET, University of Buenos Aires) which is part of two articles (Borzi and Zirattu, 2026a, Borzi and Zirattu, 2026b).

It is worth anticipating that the systems introduced here, except pure variable inclusion logics, are substructural, therefore cannot be handled with standard matrix semantics, making their semantic treatment less straightforward. Consequently, the scope of this chapter's novelties is restricted to regular companions of **CL** that are subsystems of WK logics in the sense described above, with further developments left for future work.

3.1 Pure Variable Inclusion Logics

The first contributions that developed refinements of weak variable inclusion logics and WK logics in particular are due to Paoli et al., 2021 and, for their balanced siblings, Szmuc and Rubin, 2022. The guiding motivation in Paoli et al., 2021 is the dissatisfaction with the fact that a logic's theorems will also be theorems of its weak left companion and its antitheorems will also be such for its weak right companion, which, combined with the structurality—hence the monotonicity—of these logics, determines exceptions to variable inclusion. These exceptions are particularly problematic when such systems are interpreted through the lenses of relevance. For instance, **PWK** cannot rule out arguments as Example Ia., while $\mathbf{K_3^w}$ cannot reject arguments as Example Ib., which, as it was discussed in section 1.1, constitute prime examples of irrelevance.

Notice that, if structurality is not assumed, in an account of content inclusion understood from right to left, the mere fact that a system contains antitheorems does not, by itself, violate the variable-inclusion constraint, since the empty set on the right is trivially contained in any set of premises. Although one might be unsure how to assess an argument with contradictory premises and no conclusion[1], from the standpoint of topic or content inclusion the issue becomes much clearer when a system allows any conclusion to be added at will. Similarly, when content inclusion is considered from left to right, the mere presence of theorems does not in itself constitute an exception to variable containment, since the empty set on the left is, again, trivially included in any set of conclusions. As in the case of antitheorems, such situations may not seem to exhibit obvious irrelevance; yet the problem arises when arbitrary premises can be introduced that bear no relation to the conclusion. Both of these undesirable situations[2] stem from monotonicity——specifically, monotonicity on the right in the first case, and on the left in the second.

The proposal in Paoli et al., 2021 may be viewed as a response to this issue, though constrained to structural systems. Within this constraint, the only solution to the problem is to remove antitheorems or theorems altogether from the weak right or left companions of a logic, respectively. Namely, in the first case not only is the inference from $\psi\wedge\neg\psi$ to an arbitrary φ blocked, but so is an inference with no conclusion and $\psi\wedge\neg\psi$ as premise. Likewise, in the second case both the inference from any ψ to $\varphi\vee\neg\varphi$ and that with no premise and $\varphi\vee\neg\varphi$ as conclusion are rejected.

[1]This point is controversial not only from the point of view of relevance, but also from a more general perspective on consequence. Notice however that discussions around this issue are mostly in relation to what contradictions may imply, rather than just considering what happens when premises are contradictory but conclusions are empty. For instance, in Priest, 1999 the three main options on the market are listed as the total, partial and null view of negation, corresponding, respectively, to the case in which contradictions are explosive, imply something but not everything, do not imply anything. These three accounts are further evaluated in Iacona, 2026 with attention to the notion of relevance.

[2]However, notice that both in the case of contradictions and tautologies the problems for relevance only arise under an account which takes such statements to have *some* content, or to be about *something* and not about *everything*. These limit cases warrant further investigation, which is deferred to future work.

Hence in Paoli et al., 2021 new systems are introduced, called *pure* variable inclusion logics, which, given a structural logic **L**, are subsystems of the weak right or left companion of **L** that are characterizable as in Definitions 2.4.1 and 2.4.2 with the elimination of clause (i). Namely:

Definition 3.1.1. Given a logic **L**, its *pure right companion* is the logic $\mathbf{L^{pr}}$ characterized as follows:

$$\Gamma \vdash_{\mathbf{L^{pr}}} \Delta \quad \text{iff} \quad \Gamma \vdash_{\mathbf{L}} \Delta \quad \text{and} \quad \exists \Delta' \mid \Delta' \subseteq \Delta \text{ and } \Delta' \neq \varnothing, \\ \text{s.t. } \Gamma \vdash_{\mathbf{L}} \Delta' \text{ and } Var(\Delta') \subseteq Var(\Gamma)$$

Definition 3.1.2. Given a logic **L**, its *pure left companion* is the logic $\mathbf{L^{pl}}$ characterized as follows:

$$\Gamma \vdash_{\mathbf{L^{pl}}} \Delta \quad \text{iff} \quad \Gamma \vdash_{\mathbf{L}} \Delta \quad \text{and} \quad \exists \Gamma' \mid \Gamma' \subseteq \Gamma \text{ and } \Gamma' \neq \varnothing, \\ \text{s.t. } \Gamma' \vdash_{\mathbf{L}} \Delta \text{ and } Var(\Gamma') \subseteq Var(\Delta)$$

Accordingly, Szmuc and Rubin, 2022 define the pure *refined* companions of a logic with the analogous modifications of Definitions 2.4.3 and 2.4.4. Keeping the same terminology, such systems will be called pure *balanced* logics, and are defined as follows:

Definition 3.1.3. Given a logic **L**, its *pure balanced right companion* is the logic $\mathbf{L^{pr\pm}}$ characterized as follows:

$$\Gamma \vdash_{\mathbf{L^{pr\pm}}} \Delta \quad \text{iff} \quad \Gamma \vdash_{\mathbf{L}} \Delta \quad \text{and} \quad \exists \Delta' \mid \Delta' \subseteq \Delta \text{ and } \Delta' \neq \varnothing, \text{ s.t. } \Gamma \vdash_{\mathbf{L}} \Delta' \text{ and} \\ \text{both } Var^{+}(\Delta') \subseteq Var^{+}(\Gamma) \text{ and } Var^{-}(\Delta') \subseteq Var^{-}(\Gamma)$$

Definition 3.1.4. Given a logic **L**, its *pure balanced left companion* is the logic $\mathbf{L^{pl\pm}}$ characterized as follows:

$$\Gamma \vdash_{\mathbf{L^{pl\pm}}} \Delta \quad \text{iff} \quad \Gamma \vdash_{\mathbf{L}} \Delta \quad \text{and} \quad \exists \Gamma' \mid \Gamma' \subseteq \Gamma \text{ and } \Gamma' \neq \varnothing, \text{s.t. } \Gamma' \vdash_{\mathbf{L}} \Delta \text{ and} \\ \text{both } Var^{+}(\Gamma') \subseteq Var^{+}(\Delta) \text{ and } Var^{-}(\Gamma') \subseteq Var^{-}(\Delta)$$

Just as in the case of their *weak* siblings, also pure variable inclusion logics can be characterized through a general method if the system they are companions of is structural. As the previous chapter, first the construction for the regular systems will be introduced followed by the one for balanced logics. For the regular case, these notions are needed:

Definition 3.1.5. Given any logical matrix $\mathfrak{M} = \langle \mathsf{A}, D\rangle$, and any element $e \notin A$—A being the universe of A—the *m-infectious extension* of $\mathfrak{M}$ with e is the matrix $\mathfrak{M}^m[e] = \langle \mathsf{A}[e], A\rangle$, while its *n-infectious extension* is the matrix $\mathfrak{M}^n[e] = \langle \mathsf{A}[e], \{e\}\rangle$.

A purely syntactic characterization result can be proved with respect to both n and m-infectious extensions[3]:

Theorem 3.1.1. *For any matrix $\mathfrak{M}$, its m-infectious extension $\mathfrak{M}^m[e]$ is such that:*

$$\Gamma \vDash_{\mathfrak{M}^m[e]} \Delta \quad \textit{iff} \;\; \exists \Delta' \mid \Delta' \subseteq \Delta \textit{ and } \Delta' \neq \varnothing, \\ \textit{s.t.} \;\; Var(\Delta') \subseteq Var(\Gamma)$$

Theorem 3.1.2. *For any matrix $\mathfrak{M}$, its n-infectious extension $\mathfrak{M}^n[e]$ is such that:*

$$\Gamma \vDash_{\mathfrak{M}^n[e]} \Delta \quad \textit{iff} \;\; \exists \Gamma' \mid \Gamma' \subseteq \Gamma \textit{ and } \Gamma' \neq \varnothing, \\ \textit{s.t.} \;\; Var(\Gamma') \subseteq Var(\Delta)$$

Hence, Paoli et al., 2021 prove the following results[4]—which can also be proved from Theorems 2.4.1 and 3.1.2 in the first case, and Theorems 2.4.2 and 3.1.1 in the second—where, given a matrix $\mathfrak{M}$, the following bundles

[3]This is proved in Zirattu, 2022 for a specific case but the demonstration can be adapted to the general case.

[4]Both Paoli et al., 2021 and Szmuc and Rubin, 2022 only consider single conclusion inferences, but their results can be straightforwardly extended to multiple conclusions. Moreover, just as in the case of weak variable inclusion logics, the results in Paoli et al., 2021 use an alternative but equivalent construction which is obtained via Płonka sums.

$\mathfrak{M}^{pl} = \{\mathfrak{M}^l[e], \mathfrak{M}^n[e]\}$ and $\mathfrak{M}^{pr} = \{\mathfrak{M}^r[e], \mathfrak{M}^m[e]\}$ are, respectively, its pure left and pure right extensions.

Theorem 3.1.3.

$$\Gamma \vDash_{\mathfrak{M}^{pl}} \Delta \quad \textit{iff}\ \Gamma \vDash_{\mathfrak{M}} \Delta \ \ \textit{and} \ \ \exists \Gamma' \mid \Gamma' \subseteq \Gamma \ \textit{and}\ \Gamma' \neq \varnothing,$$
$$s.t.\ \ \Gamma' \vDash_{\mathfrak{M}} \Delta \ \textit{and}\ Var(\Gamma') \subseteq Var(\Delta)$$

Theorem 3.1.4.

$$\Gamma \vDash_{\mathfrak{M}^{pr}} \Delta \quad \textit{iff}\ \Gamma \vDash_{\mathfrak{M}} \Delta \ \ \textit{and} \ \ \exists \Delta' \mid \Delta' \subseteq \Delta \ \textit{and}\ \Delta' \neq \varnothing,$$
$$s.t.\ \ \Gamma \vDash_{\mathfrak{M}} \Delta' \ \textit{and}\ \ Var(\Delta') \subseteq Var(\Gamma)$$

Given that generalized Wójcicki's theorem guarantees that each structural logic can be characterized by a class of matrices, this construction provides a recipe to obtain the pure left and pure right companions of any structural logic. In fact, the general case in which a logic is characterized by a class of matrices $\mathfrak{M}$, can be covered considering the class obtained as $\bigcup \mathfrak{M}_i^{pl}$ for each $\mathfrak{M}_i \in \mathfrak{M}$ in the pure left case, $\bigcup \mathfrak{M}_i^{pr}$ for each $\mathfrak{M}_i \in \mathfrak{M}$ in the pure right case.

Notice that Theorems 3.1.1 and 3.1.2 show that the consequence relations induced by the m- and n-infectious extensions, respectively, coincide with purely syntactic constraints that lead to rather peculiar outcomes. For instance, in the former case, an inference from a disjunction to one of its disjuncts would count as valid, while in the latter, a conjunction would be validly inferred from one of its conjuncts. These unusual results are accounted for in Paoli et al., 2021 by appealing to the 'nonsense' interpretation of the infectious value discussed in section 2.5. With this interpretation, the first kind of consequence relation coincides with *preservation of meaningfulness*, while the second corresponds to *preservation of* *nonsense*—hence explaining the terminology adopted for *m*- and *n*- infectious extensions. A similar account is offered in Zirattu, 2022, where the former is interpreted as preservation of *propositionality*——that is, if the premises express a proposition, then at least some of the conclusions must do so as well——while the latter is understood as preservation of *non-propositionality*——that is, if the premises fail to express a proposition, then at least some of the conclusions must also fail to do so. Also the off-topic interpretation by Beall, 2016 introduced in section 2.5 offers a compelling intuition for this non-standard choice of designated values: in the first case one can never draw only off-topic conclusions from on-topic premises, whereas in the other, off-topic premises cannot entail conclusions that are all on-topic. Thus, in light of this discussion, pure variable inclusion logics can be understood as systems that add to some given notion of consequence the requirement of meaningfulness/propositionality/on-topicality preservation from premises to conclusions or vice-versa. This interpretation will be further discussed in the next chapter.

Paoli et al., 2021 separately discuss the case of pure left and right companions of **CL**, called $\mathbf{CL^{pr}}$ and $\mathbf{CL^{pl}}$, which, correspond, respectively, to subsystems of $\mathbf{K_3^w}$ lacking antitheorems and of **PWK** lacking theorems. Moreover, it is well known (Deutsch, 1981, Ferguson, 2015, Paoli, 2007) that $\mathbf{CL^{pr}}$ corresponds to the first-degree fragment of Parry's logic of Analytic Implication **PAI** in Parry, 1933, while $\mathbf{CL^{pl}}$ corresponds to the first-degree fragment of Epstein's logic of Dual Dependence **DD** in Epstein, 1990. A significant feature of these systems is that they cannot be characterized by a unique matrix. This can be proved showing that they do not satisfy the cancellation property.

Fact 3.1.1. Neither $\mathbf{CL^{pr}}$ nor $\mathbf{CL^{pl}}$ have the cancellation property.

Proof. In the case of $\mathbf{CL^{pr}}$, even though $p, \neg p, q \vdash_{\mathbf{CL^{pr}}} \neg q$, $p, \neg p \nvdash_{\mathbf{CL^{pr}}} r$ and $Var(p, \neg p) \cap Var(q, \neg q) = \varnothing$, the premises $\{p, \neg p\}$ cannot be canceled or else the inference fails, *i.e* $q \nvdash_{\mathbf{CL^{pr}}} \neg q$. Similarly, $q \vdash_{\mathbf{CL^{pl}}} \neg q, p, \neg p$, but $q \nvdash_{\mathbf{CL^{pl}}} \neg q$. □

The counterexample for $\mathbf{CL^{pr}}$ is the same as in Paoli et al., 2021 since it also applies to the single conclusion framework they consider, while no counterexample can be found for $\mathbf{CL^{pl}}$ in that setting given that restricting the number of conclusions allows to characterize $\mathbf{CL^{pl}}$ via a single matrix. Similarly, if only single-premise inferences are considered, no counterexample can be found for $\mathbf{CL^{pr}}$, as it is shown in Szmuc, 2021 where a characteristic matrix is presented.

In Paoli et al., 2021, $\mathbf{CL^{pl}}$ and $\mathbf{CL^{pr}}$ are also presented as consequence relations based on an order over the algebra WK. Since WK is a bisemilattice, there are two distinct bisemilattice orders (see the previous chapter) associated to the meet and join, that is, $\leq^{\wedge}_{\mathbf{WK}}$ and $\leq^{\vee}_{\mathbf{WK}}$ respectively, as it is displayed in Figure 2.3. Each of these can then be used to induce a different consequence relation, as it is proved in Theorem 3.5 of Paoli et al., 2021[5]:

Fact 3.1.2.

$$\Gamma \vdash_{\mathbf{CL^{pl}}} \Delta \quad \text{iff} \quad \text{for every } v \in \mathrm{Hom}_{\mathsf{WK}} : \\ \mathtt{min}^{\vee}\{v(A) \mid A \in \Gamma\} \leq^{\vee}_{\mathsf{WK}} \mathtt{max}^{\vee}\{v(B) \mid B \in \Delta\}$$

[5]In Paoli et al., 2021 this is proved for single conclusion inferences but the generalization to multiple conclusions can be straightforwardly adapted from their proof.

Fact 3.1.3.

$$\Gamma \vdash_{\mathbf{CL^{pr}}} \Delta \quad \text{iff} \quad \text{for every } v \in \mathrm{Hom}_{\mathsf{WK}}:$$
$$\mathtt{min}^{\wedge}\{v(A) \mid A \in \Gamma\} \leq^{\wedge}_{\mathsf{WK}} \mathtt{max}^{\wedge}\{v(B) \mid B \in \Delta\}$$

Notice that, as it was remarked in Fact 2.3.2, the Sobociński operators $\vee_{\mathbf{S}}$ and $\wedge_{\mathbf{S}}$ coincide with $\mathtt{max}^{\wedge}$ and $\mathtt{min}^{\vee}$, respectively. For this reason, if finite sets $\Gamma, \Delta \in Fml(\mathscr{L})$ are considered, the above characterizations can be restated as follows:

$$\Gamma \vDash_{\mathbf{CL^{pl}}} \Delta \text{ iff for all } v \in \mathrm{Hom}_{\mathsf{WK}} : v(\bigwedge_{\mathsf{S}} \Gamma) \leq^{\vee}_{\mathsf{WK}} v(\bigvee_{\mathsf{WK}} \Delta)$$

$$\Gamma \vDash_{\mathbf{CL^{pr}}} \Delta \text{ iff for all } v \in \mathrm{Hom}_{\mathsf{WK}} : v(\bigwedge_{\mathsf{WK}} \Gamma) \leq^{\wedge}_{\mathsf{WK}} v(\bigvee_{\mathsf{S}} \Delta)$$

Moving on to the balanced case, the construction for such pure systems is provided in Szmuc and Rubin, 2022. The authors only preset the construction to obtain pure balanced right companions, but it can be adapted to pure balanced left companions following the same strategy as in Paoli et al., 2021, this time in a framework where negation is transformative—as discussed in section 2.5. For the first, a given notion of consequence is equipped with preservation of meaningfulness from left to right, in the other direction for the latter.

Recall Definition 2.4.7 introducing balanced left and right extensions of a matrix $\mathfrak{M}$, namely $\mathfrak{M}^l[\pm]$ and $\mathfrak{M}^r[\pm]$ respectively, which can also be extended to classes of matrices. Now, consider the following definition, analogous to Definition 3.1.5 (and Definition 14 in Szmuc and Rubin, 2022):

Definition 3.1.6. Given any logical matrix $\mathfrak{M} = \langle \mathsf{A}, D \rangle$—the *balanced-m-infectious extension* of $\mathfrak{M}$ is the matrix $\mathfrak{M}^m[\pm] = \langle \mathsf{A}[\pm], A[+] \rangle$, while its *balanced-n-infectious extension* is the matrix $\mathfrak{M}^n[\pm] = \langle \mathbf{A}[\pm], \overline{A[+]} \rangle$.

The logical consequences induced by these matrices can be characterized syntactically, as in Theorems 3.1.1 and 3.1.2, in the following manner, restricting the proof of Theorem 8 in Szmuc and Rubin, 2022 for the former, dualizing it for the latter:

Theorem 3.1.5. *For any matrix $\mathfrak{M}$, its balanced-m-infectious extension $\mathfrak{M}^m[\pm]$ is such that:*

$$\Gamma \vDash_{\mathfrak{M}^m[\pm]} \Delta \quad \textit{iff} \;\; \exists \Delta' \mid \Delta' \subseteq \Delta \textit{ and } \Delta' \neq \varnothing, \\ \textit{s.t.} \;\; Var^+(\Delta') \subseteq Var^+(\Gamma) \textit{ and } Var^-(\Delta') \subseteq Var^-(\Gamma)$$

Theorem 3.1.6. *For any matrix $\mathfrak{M}$, its balanced-n-infectious extension $\mathfrak{M}^n[\pm]$ is such that:*

$$\Gamma \vDash_{\mathfrak{M}^n[\pm]} \Delta \quad \textit{iff} \;\; \exists \Gamma' \mid \Gamma' \subseteq \Gamma \textit{ and } \Gamma' \neq \varnothing, \\ \textit{s.t.} \;\; Var^+(\Gamma') \subseteq Var^+(\Delta) \textit{ and } Var^-(\Gamma') \subseteq Var^-(\Delta)$$

In the following results—the latter being Theorem 9 in Szmuc and Rubin, 2022 extended to multiple conclusion inferences—given a matrix $\mathfrak{M}$, the following bundles $\boldsymbol{\mathfrak{M}}^{pl\pm} = \{\mathfrak{M}^l[\pm], \mathfrak{M}^n[\pm]\}$ and $\boldsymbol{\mathfrak{M}}^{pr\pm} = \{\mathfrak{M}^r[\pm]$, $\mathfrak{M}^m[\pm\}$ are, respectively, its pure balanced left and pure balanced right extensions. Both results can also be proved from Theorems 2.4.4 and 3.1.6 in the first case, and Theorems 2.4.3 and 3.1.5 in the second.

Theorem 3.1.7.

$$\Gamma \vDash_{\boldsymbol{\mathfrak{M}}^{pl\pm}} \Delta \quad \textit{iff} \; \Gamma \vDash_{\mathfrak{M}} \Delta \;\; \textit{and} \;\; \exists \Gamma' \mid \Gamma' \subseteq \Gamma \textit{ and } \Gamma' \neq \varnothing, \textit{s.t.} \; \Gamma' \vDash_{\mathfrak{M}} \Delta \textit{ and} \\ \textit{both} \;\; Var^+(\Gamma') \subseteq Var^+(\Delta) \textit{ and } Var^-(\Gamma') \subseteq Var^-(\Delta)$$

Theorem 3.1.8.

$$\Gamma \vDash_{\mathfrak{M}^{pr\pm}} \Delta \quad \textit{iff}\ \Gamma \vDash_{\mathfrak{M}} \Delta \ \ \textit{and}\ \ \exists \Delta' \mid \Delta' \subseteq \Delta \ \textit{and}\ \Delta' \neq \varnothing, s.t.\ \Gamma \vDash_{\mathfrak{M}} \Delta' \ \textit{and}$$
$$\textit{both}\ \ Var^{+}(\Delta') \subseteq Var^{+}(\Gamma)\ \textit{and}\ Var^{-}(\Delta') \subseteq Var^{-}(\Gamma)$$

As above, given that generalized Wójcicki's theorem guarantees that each structural logic can be characterized by a class of matrices, this construction can be extended as to obtain the pure balanced left and pure balanced right companions of any structural logic. In fact, the general case in which a logic is characterized by a class of matrices $\mathfrak{M}$, can be covered considering the class obtained as $\bigcup \mathfrak{M}_i^{pl\pm}$ for each $\mathfrak{M}_i \in \mathfrak{M}$ in the pure balanced left case, $\bigcup \mathfrak{M}_i^{pr\pm}$ for each $\mathfrak{M}_i \in \mathfrak{M}$ in the pure balanced right case.

As in the case of pure right and left companions of **CL**, its pure balanced companions cannot be characterized by a unique matrix in a multiple premise-multiple conclusion framework:

Fact 3.1.4. Neither $\mathbf{CL^{pr\pm}}$ nor $\mathbf{CL^{pl\pm}}$ have the cancellation property.

Proof. In the case of $\mathbf{CL^{pr\pm}}$, even though $p, \neg p, q \vee s \vdash_{\mathbf{CL^{pr\pm}}} q$ and both $p, \neg p \nvdash_{\mathbf{CL^{pr\pm}}} r$ and $Var(p, \neg p) \cap Var(q \vee r, q) = \varnothing$, the premises $\{p, \neg p\}$ cannot be canceled or else the inference fails, *i.e* $q \vee s \nvdash_{\mathbf{CL^{pr\pm}}} q$. Similarly, $q \vdash_{\mathbf{CL^{pl\pm}}} q \wedge s, p, \neg p$, but $q \nvdash_{\mathbf{CL^{pl\pm}}} q \wedge s$. □

Finally, connecting this section to the end of section 2.3 and completing the facts listed there, the following equivalences hold as a consequence of the results above and the fact that **LP** does not have antitheorems and $\mathbf{K_3}$ does not have theorems, while **BD** has none[6]:

[6] For the single-conclusion case, the lattice of all extensions of $\mathbf{CL^{Pl}}$ has been completely described in Paoli and Pra Baldi, 2024.

$$\mathbf{CL}^{\mathbf{wr}} = \mathbf{K}_{\mathbf{3}}^{\mathbf{w}} \quad \mathbf{CL}^{\mathbf{wl}} = \mathbf{PWK} \quad \mathbf{CL}^{\mathbf{pr}} = \mathbf{PAI}_{\mathtt{fde}} \quad \mathbf{CL}^{\mathbf{pl}} = \mathbf{DD}_{\mathtt{fde}}$$

$$\mathbf{K}_3^{\mathbf{wr}} = \mathbf{S}_{\mathbf{etl}} \quad \mathbf{K}_3^{\mathbf{wl}} = \mathbf{K}_3^{\mathbf{pl}} \quad \mathbf{K_3}^{\mathbf{wl}\pm} = \mathbf{K_3}^{\mathbf{pl}\pm}$$

$$\mathbf{LP}^{\mathbf{wr}} = \mathbf{LP}^{\mathbf{pr}} = \mathbf{S}_{\mathtt{fde}} \quad \mathbf{LP}^{\mathbf{wl}} = \mathbf{S}_{\mathbf{nfl}} \quad \mathbf{LP}^{\mathbf{wr}\pm} = \mathbf{LP}^{\mathbf{pr}\pm}$$

$$\mathbf{BD}^{\mathbf{wr}} = \mathbf{BD}^{\mathbf{pr}} = \mathbf{S}^{*}_{\mathtt{fde}} \quad \mathbf{BD}^{\mathbf{wl}} = \mathbf{BD}^{\mathbf{pl}}$$

$$\mathbf{BD}^{\mathbf{wr}\pm} = \mathbf{BD}^{\mathbf{pr}\pm} = \mathbf{AC}_{\mathtt{fde}} \quad \mathbf{BD}^{\mathbf{wl}\pm} = \mathbf{BD}^{\mathbf{pl}\pm}$$

3.2 Universal Variable Inclusion Logics

As it was remarked above, the *pure* route in Paoli et al., 2021 and Szmuc and Rubin, 2022 may be seen as addressing the fact that weak companions of a logic may still validate prime examples of irrelevance where theorems are implied by anything and antitheorems imply anything, while remaining in a *structural* framework. In fact, the only way to avoid such cases and at the same time maintain monotonicity of the consequence relation, is to remove theorems or antitheorems altogether.

Remaining in a structural framework has its advantages as it allows for a more or less standard semantic study, but also because it does not deviate from the most widely accepted notion of logical consequence as complying with some basic requirements, such as reflexivity, transitivity and monotonicity. However, several motivations for departing from this notion have been developed and inspired logical traditions—see, e.g., Paoli, 2002 for an overview. In particular, the relevance tradition has long expressed criticism toward monotonicity. For instance, within the more proof-theoretic approach to relevance, according to which premises are relevant to the conclusion only when they are actually *used* in its derivation, monotonicity constitutes a clear violation: a statement ψ derived from a set of premises Γ can also be derived from any set Σ containing Γ, even though the extra premises play no role in the proof.

For the purposes of the present work, there are at least two main issues imputable to monotonicity, hence at least two reasons to abandon it for content inclusion logics—or, more precisely, to restrict it. One is related to inferences with theorems or antitheorems, already partly discussed above and to be continued later, that is, clause (i) in the characterization of weak variable companions[7]. The other pertains the variable inclusion requirement for such logics, namely the condition expressed in clause (ii). This issue will be discussed in this section where the proposed solution will be motivated in line with the interpretation of containment logics as logics of content inclusion, but also from other points of view, some of which have been introduced in section 2.5.

The systems considered in this section were introduced in Borzi and Zirattu, 2026b as Uniform logics, motivated by other issues which have nothing to do with relevance. Thus, the aim of this section is to present them and discuss their interpretation as content inclusion logics. Although the approach for these logics is not as general as that in Paoli et al., 2021, it is able to cover a wide range of cases. In this work, such systems are called *universal variable inclusion companions*, given their definition:

Definition 3.2.1. Given a logic **L**, its *universal left companion* is the logic

[7]It is worth mentioning that the issue concerning weak right variable inclusion logics and the fact that contradictions entail anything, has also been addressed by Ferguson, 2015 where a non-reflexive solution is proposed. Taking inspiration from the connexive tradition and the null-account of contradictions in Priest, 1999, the author proposes a notion of containment where contradictions do not entail anything, not even themselves, thus blocking the examples of irrelevance admitted by the weak companions. Other non-reflexive versions of containment have been explored in the literature, see *e.g.* Weiss, 2019, though this route deserves a more careful consideration.

$\mathbf{uL^l}$ characterized as follows:

$$\Gamma \vdash_{\mathbf{uL^l}} \Delta \text{ iff } \Gamma \vdash_{\mathbf{L}} \Delta \text{ and } \begin{cases} \text{(i) } \varnothing \vdash_{\mathbf{L}} \Delta \text{ or} \\ \text{(ii) } \forall \Gamma' \subseteq \Gamma, Var(\Gamma') \subseteq Var(\Delta) \end{cases}$$

Definition 3.2.2. Given a logic **L**, its *universal right companion* is the logic $\mathbf{uL^r}$ characterized as follows:

$$\Gamma \vdash_{\mathbf{uL^r}} \Delta \text{ iff } \Gamma \vdash_{\mathbf{L}} \Delta \text{ and } \begin{cases} \text{(i) } \Gamma \vdash_{\mathbf{L}} \varnothing \text{ or} \\ \text{(ii) } \forall \Delta' \subseteq \Delta, Var(\Delta') \subseteq Var(\Gamma) \end{cases}$$

Both definitions resemble Definitions 2.4.2 and 2.4.1, respectively, but, as in the case of pure companions, there is one modification, this time involving clause (ii). Namely, the variable inclusion requirement in each case is replaced with the corresponding universal version: $\mathbf{LVI}_{\exists}$ is replaced by $\mathbf{LVI}_{\forall}$, while $\mathbf{RVI}_{\exists}$ is replaced by $\mathbf{RVI}_{\forall}$. Clause (i), instead, is kept as it is and, therefore, if the starting logic is structural and it has theorems or antitheorems, then its uniform left companion will share the same theorems which are implied by any formula and, similarly, its right uniform companion will keep the antitheorems that imply anything.

For this reason, universal companions may be subject the same kind of criticism that motivated the development of pure variable inclusion logics. There may be one way to resist it, briefly mentioned in the previous section, which would also account for the exceptions of variable inclusion in weak companions. This consists in regarding both theorematic and antitheorematic statements or sets of statements as being about *everything*[8], making the addition of arbitrary premises in the presence of theorems compatible

[8]Notice, however, that while this characterization of the content of theorems aligns with the standard view in possible worlds semantics, it departs from the usual treatment of antitheorems, which are typically regarded as devoid of content in the same framework.

with the condition of topic inclusion from left to right and, dually, arbitrary conclusions following from antitheorems being consistent with topic inclusion from right to left. In any case, given that this kind of inferences were taken to be prime examples of irrelevance, it is not as easy to reconcile this thesis with the core intuitions about meaning connections and more justification would be needed.

Besides this, the most distinctive and original feature of these systems is given by the change in clause (ii), now expressing what was described in section 2.1 as the most straightforward formalization of **RVI** and **LVI**, that is, plain topic inclusion. The reason to make this move as to provide a better account of content inclusion lies in the fact that, leaving aside the cases with theorems or antitheorems, while weak and pure companions perform equally well in the characterization of topic inclusion for inferences with a single premise and a single conclusion, they do not perform as well in the more general framework of multiple premises and multiple conclusions inferences. To see this, an illustrative example is given by the companions of **CL**. Recall that both $\mathbf{K_3^w}$ and $\mathbf{CL^{pr}}$ are able to discard inferences such as that in Example IIa., that is, instances of disjunction introduction, while **PWK** and $\mathbf{CL^{pl}}$ rule out inferences such as Example IIb., that is, instances of conjunction elimination. The criterion for such failures was, in the former, blocking the possibility of conclusions adding some topic which could exceed that of the premises, in the latter, avoiding the possibility of some premise being irrelevant for the conclusion. However, something peculiar happens for these systems if the analogous to Addition and Simplification are considered when multiple conclusions, for the first, or multiple premises, for the second, are allowed:

IIIa. *The Moon is made of green cheese. Hence, it is raining in Turin. Alternatively, the Moon is made of green cheese.*

IIIb. *It is raining in Turin. Moreover, the Moon is made of green cheese. Hence, it is raining in Turin.*

These examples are nothing but the reformulation of the arguments in Examples IIa. and IIb. respectively, when multiple conclusions are understood disjunctively and when multiple premises are read conjunctively. Moreover, just as in the single premise-single conclusion framework, the above arguments constitute exceptions to the proviso of content inclusion either from conclusions to premises or vice versa. However, none of the systems seen so far invalidates Example IIIa. nor IIIb.—in particular, $\mathbf{K_3^w}$ and $\mathbf{CL^{pr}}$, in which IIa. fails as a violation of right-to-left content inclusion, and **PWK** and $\mathbf{CL^{pl}}$, that rule out IIb. as a violation of left-to-right content inclusion. As it was mentioned in section 2.3, one way to explain this mismatch is precisely clause (ii) in the characterization of such systems, as it only requires the variables of *some* of the conclusions or premises to be contained in the other side. In fact, once variable inclusion is ensured for that nonempty subset, then any additional—potentially totally unrelated—formula may just be added on either side of the inference.

However, it seems that if one accepts content inclusion as a reason to reject Addition or Simplification, then the same motivation should lead to the rejection of monotonicity. In particular, if *all* of the conclusions are to be relevantly entailed from the premises, then it should not be possible to add just *any* formula on the right, while if *all* premises should be relevant for the conclusions, then it should not be allowed to add just *any* other statement on the left. Replacing $\mathbf{RVI}_\exists$ for $\mathbf{RVI}_\forall$ and $\mathbf{LVI}_\exists$ for $\mathbf{LVI}_\forall$, precisely ensure this outcome.

To see the effect of this modification, consider the universal variable inclusion companions of **CL**, $\mathbf{uCL^l}$ and $\mathbf{uCL^r}$, obtained as in Definitions 3.2.1 and 3.2.2 respectively. Their consequence relation can be semantically characterized over the algebra WK as follows (as it is shown in Borzi and Zirattu, 2026b)

Theorem 3.2.1. $\Gamma \vDash_{\mathbf{uCL^l}} \Delta$ *iff for every valuation* $v \in Hom_{\mathsf{WK}}$*: if* $\mathtt{min}^{\wedge_{\mathsf{WK}}}\{v(\gamma) \mid \gamma \in \Gamma\} \neq f$*, then* $\mathtt{max}^{\vee_{\mathsf{WK}}}\{v(\delta) \mid \delta \in \Delta\} \neq f$*.*

Proof. Both directions are proved by contraposition:

(⇒) Suppose that there is a valuation $v \in \mathrm{Hom}_{\mathsf{WK}}$ such that $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} \in \{t, e\}$ but $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = f$. So for all $\delta \in \Delta, v(\delta) = f$, thus by Fact 2.3.1, for all $p \in Var(\Delta)$, $v(p) \in \{t, f\}$, which guarantees the existence of a valuation $v' \in \mathrm{Hom}_{\mathsf{B}_2}$ such that $v'(\delta) = f$ for all $\delta \in \Delta$, hence that $\varnothing \nvDash_{\mathbf{CL}} \Delta$. Now, if $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = t$, then for all $\gamma \in \Gamma, v(\gamma) = t$ and, by Fact 2.3.1, there is no $q \in Var(\Gamma)$ such that $v(q) = e$. Then it is possible to construct a valuation $v'' \in \mathrm{Hom}_{\mathsf{B}_2}$ assigning the same value as v to each variable in Γ, Δ and any other value, say t, to the others. Hence v'' is such that $\Gamma \nvDash_{\mathbf{CL}} \Delta$. If, instead, $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = e$, then there is $\gamma \in \Gamma$ such that $v(\gamma) = e$ and by Fact 2.3.1 there is a $q \in Var(\Gamma)$ such that $v(q) = e$. But since by the first assumption for all $p \in Var(\Delta)$, $v(p) \in \{t, f\}$, it follows that $q \notin Var(\Delta)$. Therefore, $Var(\Gamma) \nsubseteq Var(\Delta)$.

(⇐) Assume either that $\Gamma \nvDash_{\mathbf{CL}} \Delta$ or that both $\varnothing \nvDash_{\mathbf{CL}} \Delta$ and $Var(\Gamma) \nsubseteq Var(\Delta)$. In the first case, there is $v \in \mathrm{Hom}_{\mathsf{B}_2}$ such that $v(\gamma) = t$ for all $\gamma \in \Gamma$ but $v(\delta) = f$ for all $\delta \in \Delta$, but given that $v \in \mathrm{Hom}_{\mathsf{WK}}$ and that $\mathtt{min}^{\wedge}\{v(\gamma)|\gamma \in \Gamma\} = t$ while $\mathtt{max}^{\vee}\{v(\delta)|\delta \in \Delta\} = f$, $\Gamma \nvDash_{\mathbf{uCL^l}} \Delta$.

In the second case, there is a valuation $v \in \mathrm{Hom}_{\mathsf{B}_2}$ such that $v(\delta) = f$ for all $\delta \in \Delta$. Moreover, by the assumption that $Var(\Gamma) \nsubseteq Var(\Delta)$, there is some $p \in Var(\Gamma)$ such that $p \notin Var(\Delta)$. Now construct a valuation $v' \in \mathrm{Hom}_{\mathsf{WK}}$ such that for all variables q, $v'(q) = v(q)$ if $q \in Var(\Delta)$ and $v'(q) = e$ otherwise. By Fact 2.3.1, this entails that for some $\gamma \in \Gamma$, $v(\gamma) = e$. Therefore, $\mathtt{min}^{\wedge}\{v(\gamma)|\gamma \in \Gamma\} = e$ while $\mathtt{max}^{\vee}\{v(\delta)|\delta \in \Delta\} = f$.

□

Theorem 3.2.2. $\Gamma \vDash_{\mathbf{uCL^r}} \Delta$ *iff for every valuation* $v \in$ *Hom*$_{\mathsf{WK}}$*: if* $\mathtt{min}^{\wedge}\{v(\gamma) \,|\, \gamma \in \Gamma\} = t$*, then* $\mathtt{max}^{\vee}\{v(\delta) \,|\, \delta \in \Delta\} = t$*.*

Proof. ($\Rightarrow$) Suppose that there is a valuation $v \in \mathrm{Hom}_{\mathsf{WK}}$ such that $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = t$ but $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} \in \{f, e\}$. So for all $\gamma \in \Gamma, v(\Gamma) = t$, thus by Fact 2.3.1, for all $p \in Var(\Gamma)$, $v(p) \in \{t, f\}$, which guarantees the existence of a valuation $v' \in \mathrm{Hom}_{\mathsf{B}_2}$ such that $v'(\gamma) = t$ for all $\gamma \in \Gamma$, hence that $\Gamma \nvDash_{\mathbf{CL}} \varnothing$. Now, if $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = f$, then for all $\delta \in \Delta, v(\delta) = f$ and, by Fact 2.3.1, there is no $q \in Var(\Delta)$ such that $v(q) = e$. Then it is possible to construct a valuation $v'' \in \mathrm{Hom}_{\mathsf{B}_2}$ assigning the same value as v to each variable in Γ, Δ and any other value, say t, to the others. Hence v'' is such that $\Gamma \nvDash_{\mathbf{CL}} \Delta$. If, instead, $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = e$, then there is $\delta \in \Delta$ such that $v(\delta) = e$ and by Fact 2.3.1 there is a $q \in Var(\Delta)$ such that $v(q) = e$. But since by the first assumption for all $p \in Var(\Gamma)$, $v(p) \in \{t, f\}$, it follows that $q \notin Var(\Gamma)$. Therefore, $Var(\Delta) \nsubseteq Var(\Gamma)$.

($\Leftarrow$) Assume either that $\Gamma \nvDash_{\mathbf{CL}} \Delta$ or that both $\Gamma \nvDash_{\mathbf{CL}} \varnothing$ and $Var(\Delta) \nsubseteq$

$Var(\Gamma)$. In the first case, there is $v \in \text{Hom}_{\mathsf{B_2}}$ such that $v(\gamma) = t$ for all $\gamma \in \Gamma$ but $v(\delta) = f$ for all $\delta \in \Delta$, but given that $v \in \text{Hom}_{\mathsf{WK}}$ and that $\texttt{min}^{\wedge}\{v(\gamma)|\gamma \in \Gamma\} = t$ while $\texttt{max}^{\vee}\{v(\delta)|\delta \in \Delta\} = f$, $\Gamma \not\vDash_{\mathbf{uCL^r}} \Delta$.

In the second case, there is a valuation $v \in \text{Hom}_{\mathsf{B_2}}$ such that $v(\gamma) = t$ for all $\gamma \in \Gamma$. Moreover, by the assumption that $Var(\Delta) \nsubseteq Var(\Gamma)$, there is some $p \in Var(\Delta)$ such that $p \notin Var(\Gamma)$. Now construct a valuation $v' \in \text{Hom}_{\mathsf{WK}}$ such that for all variables q, $v'(q) = v(q)$ if $q \in Var(\Gamma)$ and $v'(q) = e$ otherwise. By Fact 2.3.1, this entails that for some $\delta \in \Delta$, $v(\delta) = e$. Therefore, $\texttt{max}^{\vee}\{v(\delta)|\delta \in \Delta\} = e$ while $\texttt{min}^{\wedge}\{v(\gamma)|\gamma \in \Gamma\} = t$.

□

For finite Γ and Δ, these characterizations coincide with the following equivalence, which also shows that $\mathbf{uCL^r}$ and $\mathbf{uCL^l}$ are subsystems of $\mathbf{K_3^w}$ and **PWK**, respectively, where formulas are aggregated in accordance with the WK-conjunction on the left and the WK-disjunction on the right[9]:

$$\Gamma \vDash_{\mathbf{uCL^l}} \Delta \ \text{ iff } \ \bigwedge_{\mathsf{WK}}\Gamma \vDash_{\mathbf{uCL^l}} \bigvee_{\mathsf{WK}}\Delta \ \text{ iff } \ \bigwedge_{\mathsf{WK}}\Gamma \vDash_{\mathbf{PWK}} \bigvee_{\mathsf{WK}}\Delta$$

$$\Gamma \vDash_{\mathbf{uCL^r}} \Delta \ \text{ iff } \ \bigwedge_{\mathsf{WK}}\Gamma \vDash_{\mathbf{uCL^r}} \bigvee_{\mathsf{WK}}\Delta \ \text{ iff } \ \bigwedge_{\mathsf{WK}}\Gamma \vDash_{\mathbf{K_3^w}} \bigvee_{\mathsf{WK}}\Delta$$

An immediate consequence is the fact that neither $\mathbf{uCL^r}$ nor $\mathbf{uCL^l}$ are

[9]As a reviewer pointed out, a potential objection to this non-monotonic approach runs as follows. One might argue that the ability of weak and pure companions—unlike their universal siblings—to distinguish between the two ways of combining formulas (monotonic and non-monotonic) is a virtue insofar as it yields greater expressive power. While this feature may be desirable for certain purposes, much as many non-classical logics make such a distinction explicit even at the object-language level, the following pages will argue that, both from the standpoint of content inclusion and on independent grounds—such as the motivations for the infectious value discussed in section 2.5—there are compelling reasons to treat the object-linguistic and meta-linguistic ways of combining formulas uniformly.

closed under monotonicity, thus that they are substructural. Indeed:

$$p \vDash_{\mathbf{uCL^r}} p \text{ but } p \nvDash_{\mathbf{uCL^r}} p, q \qquad\qquad p \vDash_{\mathbf{uCL^l}} p \text{ but } p, q \nvDash_{\mathbf{uCL^l}} p$$

Given these characterizations, it is now easier to compare the universal companions of **CL** with both its weak and pure companions. More specifically, the above considerations suggest that $\mathbf{uCL^l}$ offers a more adequate criterion of relevance from premises to conclusions, ensuring that premises are *entirely* (rather than partially) relevant to the conclusions drawn. And, dually, $\mathbf{uCL^r}$ ensures that the subject-matter of *all* conclusions (and not just *some* of them) is included in that of the premises. While this level of relevance is maintained in **PWK** and $\mathbf{CL^{pl}}$ on the one hand, and $\mathbf{K_3^w}$ and $\mathbf{CL^{pr}}$ on the other hand, only for single premise/conclusion inferences respectively, $\mathbf{uCL^l}$ and $\mathbf{uCL^r}$ extend it to a broader framework, allowing for multiple premises/conclusions without losing this key feature.

Notice that, although monotonicity fails in both $\mathbf{uCL^r}$ and $\mathbf{uCL^l}$—in particular, on the right for the former and on the left for the latter—Borzi and Zirattu, 2026b show that two controlled versions of monotonicity can be recaptured for both. One is the case in which an inference has theorematic conclusions or antitheorematic premises—for $\mathbf{uCL^l}$ and $\mathbf{uCL^r}$ respectively—in which case monotonicity is fully admitted as clause (i) above witnesses. The other is a consequence of clause (ii) in the characterization of these systems and it may be seen as showing that the degree of monotonicity allowed cannot violate the topical filter expressed by the syntactic constraint on variable containment:

Fact 3.2.1. If $\Gamma \vDash_{\mathbf{uCL^l}} \Delta$ and $\varnothing \nvDash_{\mathbf{uCL^l}} \Delta$, then $\Gamma', \Gamma \vDash_{\mathbf{uCL^l}} \Delta$ provided that $Var(\Gamma') \subseteq Var(\Delta)$.

If $\Gamma \vDash_{\mathbf{uCL^r}} \Delta$ and $\Gamma \nvDash_{\mathbf{uCL^r}} \varnothing$, then $\Gamma \vDash_{\mathbf{uCL^l}} \Delta, \Delta'$ provided that $Var(\Delta') \subseteq Var(\Gamma)$.

Finally, similarly to weak and pure companions, Borzi and Zirattu, 2026b introduce a relatively general way to obtain a semantic characterization of the universal right or left companion of a structural logic which satisfies some properties—here extended to logics which are not necessarily finitary. First, it has to be characterized by a unique matrix $\mathfrak{M} = \langle \mathsf{A}, D \rangle$ with A being of type $\mathscr{L}$ and with universe A, which has $\langle \mathsf{B}_2, \{t\} \rangle$ as a submatrix—*i.e.* subclassical—and such that either $D = \{t\}$—in which case the matrix is called $\mathfrak{M}_\mathbf{t}$ and the logic $\mathbf{L_t}$ to mark the preservation of *exact truth*–or such that $D = A \setminus \{f\}$—in which case the matrix is named $\mathfrak{M}_{nf}$ and the logic $\mathbf{L_{nf}}$ as it preserves *non-falsity.* Moreover, the orders induced by $\vee$ over A in $\mathfrak{M}_t$ and the order induced by and $\wedge$ over A in the case of $\mathfrak{M}_{nf}$ have to satisfy, respectively, the following conditions, for any set of formulas $\Sigma \subseteq Fml(\mathscr{L})$ and valuation $v \in \mathrm{Hom}_{\mathfrak{M}_t}$ or $v \in \mathrm{Hom}_{\mathfrak{M}_{nf}}$:

a. $\mathtt{max}^{\vee_\mathsf{A}}\{v(\sigma)|\sigma \in \Sigma\} = f$ if an only if for all $\sigma \in \Sigma$, $v(\sigma) = f$

$\mathtt{max}^{\vee_\mathsf{A}}\{v(\sigma)|\sigma \in \Sigma\} \neq t$ if an only if for all $\sigma \in \Sigma$, $v(\sigma) \neq t$

b. $\mathtt{min}^{\wedge_\mathsf{A}}\{v(\sigma)|\sigma \in \Sigma\} = t$ if an only if for all $\sigma \in \Sigma$, $v(\sigma) = t$

$\mathtt{min}^{\wedge_\mathsf{A}}\{v(\sigma)|\sigma \in \Sigma\} \neq f$ if an only if for all $\sigma \in \Sigma$, $v(\sigma) \neq f$

Examples of logics that satisfy these requirements are **LP** and $\mathbf{K_3}$, but not **BD**, given that, although it is subclassical and characterizable by a unique matrix, not only the associated set of designated values is different from $\{t\}$ or $\{t, b, n\}$, but also two conjuncts being not false does not entail that the conjunction will also be not false.

Now, for each $\mathbf{L_t}$ consider its universal right companion $\mathbf{uL_t^r}$ and for each $\mathbf{L_{nf}}$ its universal left companion $\mathbf{uL_{nf}^l}$, obtained according to Definitions 3.2.2 and 3.2.1 respectively. In the following, let $\mathfrak{M}_\mathfrak{t}$ and $\mathfrak{M}_{nf}$ be the matrices over an algebra A of type $\mathscr{L}$ characterizing $\mathbf{L_t}$ and $\mathbf{L_{nf}}$ respectively. Moreover, let $\mathfrak{M}_t^r[e]$ be the right infectious extension of $\mathfrak{M}_\mathfrak{t}$ and $\mathfrak{M}_{nf}^l[e]$ be the left infectious extension of $\mathfrak{M}_{\mathfrak{nf}}$—as in Definition 2.4.5. Thus, $\mathbf{uL_t^r}$ and $\mathbf{uL_{nf}^l}$ can characterized as expected (as done in Borzi and Zirattu, 2026b)—below, only the proof for the latter is outlined as the former follows the same strategy.

Theorem 3.2.3. *The universal right companion* $\mathbf{uL_t^r}$ *of* $\mathbf{L_t}$ *can be semantically characterized as follows*

$$\Gamma \vDash_{\mathbf{uL_t^r}} \Delta \textit{ if and only if for every valuation } v \in Hom_{\mathfrak{M}_t^r[e]},$$
$$\textit{if } \mathtt{min}^{\wedge_{\mathsf{A}[e]}}\{v(\gamma)|\gamma \in \Gamma\} = t \textit{ then } \mathtt{max}^{\vee_{\mathsf{A}[e]}}\{v(\delta)|\delta \in \Delta\} = t$$

Theorem 3.2.4. *The universal left companion* $\mathbf{uL_{nf}^l}$ *of* $\mathbf{L_{nf}}$ *can be semantically characterized as follows:*

$$\Gamma \vDash_{\mathbf{uL_{nf}^l}} \Delta \textit{ if and only if for every valuation } v \in Hom_{\mathfrak{M}_{nf}^l[e]},$$
$$\textit{if } \mathtt{min}^{\wedge_{\mathsf{A}[e]}}\{v(\gamma)|\gamma \in \Gamma\} \neq f \textit{ then } \mathtt{max}^{\vee_{\mathsf{A}[e]}}\{v(\delta)|\delta \in \Delta\} \neq f$$

Proof. Both directions are proved by contraposition.

($\Rightarrow$) Suppose that there is $v \in \text{Hom}_{\mathfrak{M}_{nf}^l[e]}$ such that $\mathtt{min}^{\wedge_{\mathsf{A}[e]}}\{v(\gamma)|\gamma \in \Gamma\} \neq f$ but $\mathtt{max}^{\vee_{\mathsf{A}[e]}}\{v(\delta)|\delta \in \Delta\} = f$. By this last assumption, all $\delta \in \Delta$ are such thar $v(\delta) = f$ and therefore by Fact 2.3.1 no $q \in Var(\Delta)$ is such that $v(q) = e$, meaning that one can construct $v' \in \text{Hom}_{\mathfrak{M}_{nf}}$ such that for all $\delta \in \Delta$, $v(\delta) = f$ hence $\varnothing \nvDash_{\mathfrak{M}_{nf}} \Delta$. If, moreover, $\mathtt{min}^{\wedge_{\mathsf{A}[e]}}\{v(\gamma)|\gamma \in \Gamma\} \neq e$, then by the infectiousness of e and Fact

2.3.1, there is no $p \in Var(\Gamma \cup \Delta)$ such that $v(p) = e$, hence one can construct $v'' \in \mathrm{Hom}_{\mathfrak{M}_{\mathbf{nf}}}$ such that for all variables r, $v''(r) = v(r)$ if $r \in Var(\Gamma \cup \Delta)$ and assigning any other value, say t, otherwise, so it is such that $\Gamma \not\models_{\mathbf{L_{nf}}} \Delta$. If, instead, $\mathtt{min}^{\wedge_{\mathsf{A}[e]}}\{v(\gamma)|\gamma \in \Gamma\} = e$, then there is some $\gamma \in \Gamma$ such that $v(\gamma) = e$ and again by Fact 2.3.1 there is some $p \in Var(\Gamma)$ such that $v(p) = e$. But, given that no $q \in Var(\Delta)$ is such that $v(q) = e$, then $Var(\Gamma) \not\subseteq Var(\Delta)$.

($\Leftarrow$) Assume that either $\Gamma \not\models_{\mathbf{L_{nf}}} \Delta$, or both $\varnothing \not\models_{\mathbf{L_{nf}}} \Delta$ and $Var(\Gamma) \not\subseteq Var(\Delta)$. In the first case, there is $v \in \mathrm{Hom}_{\mathfrak{M}_{nf}}$ such that $v(\gamma) \neq f$ for all $\Gamma \in \Gamma$ while $v(\delta) = f$ for all $\delta \in \Delta$. But since $v \in \mathrm{Hom}_{\mathfrak{M}^{l}_{nf}[e]}$ and $\mathtt{min}^{\wedge_{\mathsf{A}[e]}}\{v(\gamma)|\gamma \in \Gamma\} \neq f$ while $\mathtt{max}^{\vee_{\mathsf{A}[e]}}\{v(\delta)|\delta \in \Delta\} = f$, $\Gamma \not\models_{\mathbf{uL^{l}_{nf}}} \Delta$. In the second case, there is $v \in \mathrm{Hom}_{\mathfrak{M}_{nf}}$ such that $v(\delta) = f$ for all $\delta \in \Delta$, hence by Fact 2.3.1 no $p \in Var(\Delta)$ is such that $v(p) = e$. Now construct $v'' \in \mathrm{Hom}_{\mathfrak{M}^{l}_{nf}[e]}$ such that $v''(q) = v(q)$ if $q \in Var(\Delta)$ and $v(q) = e$ otherwise. By the assumption that $Var(\Gamma) \not\subseteq Var(\Delta)$, v'' will be such that for some $\gamma \in \Gamma$, $v(\gamma) = e$. Hence, $\mathtt{min}^{\wedge_{\mathsf{A}[e]}}\{v''(\gamma)|\gamma \in \Gamma\} = e$ but $\mathtt{max}^{\vee_{\mathsf{A}[e]}}\{v''(\delta)|\delta \in \Delta\} = f$.

□

From these characterization results it is straightforward to check that just as $\mathbf{uCL^r}$ and $\mathbf{uCL^l}$, the so defined universal companions are also not closed under monotonicity. In the case of $\mathbf{uL^{l}_{nf}}$, premises can only be added arbitrarily if there is a theorematic set in the conclusions, otherwise they need to respect left-to-right variable inclusion. Similarly, $\mathbf{uL^{l}_{nf}}$, allows to add any conclusion only if there is an antitheorematic set in the premises, or else they need to respect right-to-left variable inclusion.

Before closing this section, it is worth highlighting that the development of universal companions, in particular $\mathbf{uCL^l}$ and $\mathbf{uCL^r}$, may also be justified from other points of view which are independent from the demand for relevance or content inclusion, extending the motivations for WK logics presented in section 2.5—as in Borzi and Zirattu, 2026b. In particular, it can be argued that the reasons for the failure of Addition and Simplification should also lead to the rejection of monotonicity on the right and monotonicity on the left, respectively.

Recall that in the interpretation by Bochvar, 1938, the underlying intuition was that no meaningless conclusions should follow from true premises. Within a multiple premises-multiple conclusions framework, this intuition can be strengthened: from a set of true premises, no meaningless conclusion—understood as any set containing at least one meaningless statement—should be derivable. Accordingly, in a language admitting nonsense statements, not only Addition but also monotonicity on the right ought to fail. Dually, focusing now on the interpretation by Halldén, 1949, just as a false conclusion should not follow from a meaningless premise, the presence of even a single meaningless statement within the set of premises should not license deriving a set of false conclusions. Hence, not only Simplification, but also monotonicity on the left should be rejected.

Regarding the epistemic interpretation (Fitting, 1994; Szmuc, 2019), the "cut-down" and "track-down" policies can be understood as applying not only to the pooling of opinions on individual issues—that is, formulas—but also to the pooling of sets of such issues. More specifically, under a cut-down policy, the set of experts is restricted to those who have actually expressed an opinion on the entire set of issues involved in the argument, while under a track-down policy, inconsistent opinions have to be tracked for the whole set of issues. Once more, just as these interpretations offer reasons for the failure of the corresponding operational rule, they do so also for the corresponding version of monotonicity.

Also the off-topic interpretation by Beall, 2016 is easily adaptable to the more general framework. In fact, in this case the requirement of truth preservation may be regarded as blocking the derivation of any off-topic statement even when sets of conclusions are considered, thereby rejecting not only Addition but also the possibility of adding arbitrary conclusions. Finally, the computational interpretation proposed by Ferguson, 2014 can also be extended to the setting in which a computer retrieves the value of more statements taken as a set. For instance, in this context obtaining the value of φ does not guarantee that no error will be encountered when evaluating the pair $\{\varphi, \psi\}$, since the retrieval of the value of ψ may fail. Therefore, an arbitrary addition of formulas to a set may generate errors in their evaluation, motivating the rejection of monotonicity.

3.3 Universal Pure Variable Inclusion Logics

The respective objections moved by pure and universal companions with respect to their weak siblings are not only compatible, but also seem to share something beyond the mere attempt to better capture a satisfactory notion of relevance. Pure variable inclusion logics remove theorems or antitheorems which are carried over from the logic they are companions of because of the violations of relevance caused in a monotonic framework.

Universal variable inclusion logics instead, look at the effect of monotonicity on the variable inclusion requirement itself. Hence, both refinements seek a way to reduce the irrelevance caused by monotonicity. Although pure logics do so without necessarily departing from structurality, their motivation is compatible with the one guiding universal logics: if monotonicity is the problem, then why not addressing it altogether for each case of irrelevance that it produces.

There are at least two ways to combine the pure and universal spirits, both discussed in Borzi and Zirattu, 2026a. The most straightforward is just to merge the two approaches: eliminating theorems or antitheorems and at the same requiring universal variable inclusion—obtaining *universal pure companions*. This option will be outlined here, while the other will be presented in the next section.

Definition 3.3.1. Given a logic **L**, its *Universal Pure Right Variable Inclusion companion* $\mathbf{uL^{pr}}$ is the logic characterized as follows:

$$\Gamma \vdash_{\mathbf{uL^{pr}}} \Delta \;\text{ iff }\; \Gamma \vdash_{\mathbf{L}} \Delta \;\text{ and } \Delta \neq \varnothing \text{ and } \;\forall \Delta' \mid \Delta' \subseteq \Delta$$
$$\text{s.t. } \Delta' \neq \varnothing, \;\; Var(\Delta') \subseteq Var(\Gamma)$$

Definition 3.3.2. Given a logic **L**, its *Universal Pure Left Variable Inclusion companion* $\mathbf{uL^{pl}}$ is the logic characterized as follows:

$$\Gamma \vdash_{\mathbf{uL^{pl}}} \Delta \;\text{ iff }\; \Gamma \vdash_{\mathbf{L}} \Delta \;\text{ and } \Gamma \neq \varnothing \;\textbf{and}\; \;\forall \Gamma' \mid \Gamma' \subseteq \Gamma$$
$$\text{s.t. } \Gamma' \neq \varnothing, \;\; Var(\Gamma') \subseteq Var(\Delta)$$

Notice that, with respect to Definitions 2.4.2 and 2.4.1, not only clause (i) is dropped as in pure variable inclusion logics, but clause (ii) is changed, this time, with the intersection of $\mathbf{RVI}_{\exists}$ and $\mathbf{RVI}_{\forall}$ for the former and for the latter with the intersection of $\mathbf{LVI}_{\exists}$ and $\mathbf{LVI}_{\forall}$, *i.e.*, as shown by Fact

2.1.3, $\mathbf{RVI}_{\forall\varnothing}$ and $\mathbf{LVI}_{\forall\varnothing}$, respectively.

Although the limitations of the results presented above on how to construct the companion of a given logic also carry over to this case, it is easy to check that they provide a relatively general recipe to obtain the universal pure companion of a logic $\mathbf{L}$ which is characterized by a unique matrix and such that its conjunction and disjunction behave according to points a. and b. in the previous section. So let $\mathbf{L_t}$ and $\mathbf{L_{nf}}$ be as in section 3.2, respectively characterized by a matrix $\mathfrak{M}_t$ and $\mathfrak{M}_{nf}$.

Theorem 3.3.1. *The universal pure right companion* $\mathbf{uL_t^{pr}}$ *of* $\mathbf{L_t}$ *and the universal left companion* $\mathbf{uL_{nf}^{pl}}$ *of* $\mathbf{L_{nf}}$ *can be semantically characterized as follows:*

$$\begin{aligned}&\Gamma \vDash_{\mathbf{uL_t^{pr}}} \Delta \textit{ if and only if for every valuation } v \in Hom_{\mathfrak{M}_t^r[e]},\\ &\textit{if } \mathtt{min}^{\wedge_{\mathsf{A}[e]}}\{v(\gamma)|\gamma \in \Gamma\} = t \textit{ then } \mathtt{max}^{\vee_{\mathsf{A}[e]}}\{v(\delta)|\delta \in \Delta\} = t\\ &\textit{and if } v(\gamma) \neq e \textit{ for all } \gamma \in \Gamma, \textit{ then } v(\delta) \neq e \textit{ for some } \delta \in \Delta\end{aligned}$$

$$\begin{aligned}&\Gamma \vDash_{\mathbf{uL_{nf}^{pl}}} \Delta \textit{ if and only if for every valuation } v \in Hom_{\mathfrak{M}_{nf}^l[e]},\\ &\textit{if } \mathtt{min}^{\wedge_{\mathsf{A}[e]}}\{v(\gamma)|\gamma \in \Gamma\} \neq f \textit{ then } \mathtt{max}^{\vee_{\mathsf{A}[e]}}\{v(\delta)|\delta \in \Delta\} \neq f\\ &\textit{and if } v(\gamma) = e \textit{ for all } \gamma \in \Gamma, \textit{ then } v(\delta) = e \textit{ for some } \delta \in \Delta\end{aligned}$$

Proof. The first claim follows from Theorem 3.1.1 and Theorem 3.2.3. The second claim follows from Theorem 3.1.2 and Theorem 3.2.4. □

Therefore, the universal pure companions of **CL** can be captured semantically as follows:

Fact 3.3.1. Let $\mathbf{uCL^{pr}}$ and $\mathbf{uCL^{pl}}$ be the universal pure right companion and the universal left companion of **CL**, respectively. Then:

$\Gamma \vDash_{\mathbf{uCL^{pr}}} \Delta$ if and only if for every valuation $v \in \mathrm{Hom}_{\mathsf{WK}}$,

if $\mathtt{min}^{\wedge_{\mathsf{WK}}}\{v(\gamma)|\gamma \in \Gamma\} = t$ then $\mathtt{max}^{\vee_{\mathsf{WK}}}\{v(\delta)|\delta \in \Delta\} = t$

and if $v(\gamma) \neq e$ for all $\gamma \in \Gamma$, then $v(\delta) \neq e$ for some $\delta \in \Delta$

$\Gamma \vDash_{\mathbf{uCL^{pl}}} \Delta$ if and only if for every valuation $v \in \mathrm{Hom}_{\mathsf{WK}}$,

if $\mathtt{min}^{\wedge_{\mathsf{WK}}}\{v(\gamma)|\gamma \in \Gamma\} \neq f$ then $\mathtt{max}^{\vee_{\mathsf{WK}}}\{v(\delta)|\delta \in \Delta\} \neq f$

and if $v(\gamma) = e$ for all $\gamma \in \Gamma$, then $v(\delta) = e$ for some $\delta \in \Delta$

Given the characterization of the universal pure companions of **CL**, it is easy to check that they coincide with the pure companions of the same logic if single premise/single conclusion inferences are considered, since their differences only emerge when multiple formulas appear on the sides of the consequence relation. Therefore in this setting, not only $\mathbf{CL^{pr}}$, but also $\mathbf{uCL^{pr}}$. corresponds to $\mathbf{PAI_{\mathtt{fde}}}$ and both $\mathbf{CL^{pl}}$ and $\mathbf{uCL^{pl}}$ correspond to $\mathbf{DD_{\mathtt{fde}}}$.

Moreover, from the above results it is clear that $\mathbf{uCL^{pr}}$ and $\mathbf{uCL^{pl}}$ are able to solve all the problems which affected $\mathbf{K_3^w}$ and **PWK** respectively, since all cases of irrelevance as Examples Ia. and Ib., as well as failures of content inclusion such as Examples IIa. and IIb. and their multiple premises/multiple conclusions counterparts, Examples IIIa. and IIIb., respectively fail in these systems.

A point worth stressing is that, as in the case of $\mathbf{CL^{pr}}$, $\mathbf{uCL^{pr}}$ invalidates inferences with empty conclusions, even when a classical antitheorem occurs among the premises, while $\mathbf{uCL^{pl}}$—like $\mathbf{CL^{pl}}$—invalidates inferences with empty premises, even when a classical theorem occurs as conclusion. This is significant because, as noted in section 3.1, such cases do not in themselves violate the relevant variable inclusion condition. Rather, the resulting violations arise only under monotonicity. Since universal companions already reject monotonicity, it is natural to ask whether monotonicity should likewise be rejected insofar as it permits irrelevance via theorems or antitheorems.

3.4 Simple Variable Inclusion Logics

This section presents an alternative approach to the issue posed by limit cases in which an inference either lacks conclusions but includes an antitheorem among the premises, or lacks premises but includes a theorem among the conclusions. Section 1.1 introduced relevance logics as a response to the paradoxes of the material and strict conditional, such as those conditionals having an impossible antecedent or a necessary consequent. At the level of the conditional itself, however, it is meaningless to ask whether a conditional of this sort that also lacks a consequent, in the first case, or lacks an antecedent, in the second, would violate relevance—since no such conditional could exist in the first place.

This issue can only be posed at the level of consequence, where the empty set either on the left or on the right of the turnstile may be allowed. At the same time, in a structural consequence relation this possibility is not really important given that, whenever an inference holds, any formula may be freely added to either side, so if one side happens to be empty, any formula can simply fill the gap. So the point can be under discussion only when monotonicity is somehow rejected or even just restricted. As anticipated in the previous section, rejecting monotonicity opens the door to possibly different ways of restricting it as to capture a satisfactory notion of content inclusion.

As seen above, universal pure right or left variable inclusion logics reject altogether inferences such as $\psi \wedge \neg\psi \vdash \varnothing$ or $\varnothing \vdash \varphi \vee \neg\varphi$ respectively, though there is no violation of the respective variable inclusion proviso. For this reason, although the pure route does succeed in blocking the standard cases of irrelevance such as $\psi \wedge \neg\psi \vdash \varphi$ and $\psi \vdash \varphi \vee \neg\varphi$——in contrast to the universal strategy——it seems to go too far by excluding also harmless inferences involving the empty set that do respect content inclusion.

Thus, the solution proposed here is inspired by the concerns driving pure and universal logics, but takes a step back with respect to the former approach avoiding the exclusion of the innocuous cases and yet rejecting the problematic ones, by means of a suitable restriction of monotonicity. Conveniently, this restriction is simple, requiring only that inferences satisfy plain variable inclusion in the relevant direction and continue to satisfy it even when additional formulas are introduced. Logics arising from this idea—introduced in Borzi and Zirattu, 2026a—will therefore be called simple variable inclusion logics and they are defined as follows:

Definition 3.4.1. Given a logic **L**, its *Simple Right Variable Inclusion companion* $\mathbf{L^{sr}}$ is the logic characterized as follows:

$$\Gamma \vdash_{\mathbf{L^{sr}}} \Delta \quad \text{iff} \quad \Gamma \vdash_{\mathbf{L}} \Delta \quad \text{and} \quad \forall \Delta' \mid \Delta' \subseteq \Delta$$
$$Var(\Delta') \subseteq Var(\Gamma)$$

Definition 3.4.2. Given a logic **L**, its *Simple Left Variable Inclusion companion* $\mathbf{L^{sl}}$ is the logic characterized as follows:

$$\Gamma \vdash_{\mathbf{uL^{pl}}} \Delta \quad \text{iff} \quad \Gamma \vdash_{\mathbf{L}} \Delta \quad \text{and} \quad \forall \Gamma' \mid \Gamma' \subseteq \Gamma$$
$$Var(\Gamma') \subseteq Var(\Delta)$$

Comparing simple variable inclusion logics to their universal siblings, they can be seen as the result of deleting clause (i) from Definitions 3.2.1 and 3.2.2. Although this is analogous to the move from weak to pure companions, the result is different as the variable inclusion requirement expressed in clause (ii), *i.e.* $\mathbf{RVI}_\forall$ or $\mathbf{LVI}_\forall$, does not require the set of variables to be included to be non-empty. Consequently, if the original logic has antitheorems, they will also be antitheorems in its simple right companion and, if the original logic has theorems, its simple left companion will share them. However, neither case will generate exceptions to irrelevance as, whenever variables are added to the empty set, they have to respect the variable inclusion proviso.

To better grasp the differences between this case and the pure route, consider the simple variable inclusion companions of **CL**, namely $\mathbf{CL^{sr}}$ and $\mathbf{CL^{sl}}$ obtained according to Definitions 3.4.1 and 3.4.2 respectively. On the one hand, $\varphi \wedge \neg\varphi \vdash_{\mathbf{CL^{sr}}} \varnothing$—unlike $\mathbf{CL^{pr}}$ or $\mathbf{uCL^{pr}}$—and also, *e.g.*, $\varphi \wedge \neg\varphi \vdash_{\mathbf{CL^{sr}}} \varphi$, but on the other hand, in general $\varphi \wedge \neg\varphi \nvdash_{\mathbf{CL^{sr}}} \psi$ since ψ may contain some variable which is not in $Var(\varphi \wedge \neg\varphi)$. Similarly, $\varnothing \vdash_{\mathbf{CL^{sl}}} \varphi \vee \neg\varphi$—unlike $\mathbf{CL^{pl}}$ or $\mathbf{uCL^{pl}}$— and also, *e.g.*, $\varphi \vdash_{\mathbf{CL^{sl}}} \varphi \vee \neg\varphi$, but in general $\psi \nvdash_{\mathbf{CL^{sl}}} \varphi \vee \neg\varphi$ because ψ could contain some variable not appearing in $Var(\varphi \wedge \neg\varphi)$. The semantic presentation of these logics can be given as follows (as done in Borzi and Zirattu, 2026a)[10]:

Theorem 3.4.1. $\Gamma \vDash_{\mathbf{CL^{sr}}} \Delta$ *if and only if for every valuation* $v \in$ *Hom*$_{\mathsf{WK}}$: $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} \leq^{\wedge}_{\mathsf{WK}} \mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\}$

Proof. Both directions are proved by contraposition.

[10] As a reviewer suggested, an interesting direction for further investigation concerns how, given the semantic characterizations provided here, these systems relate to the logics of order discussed in Font, 2016, ch.7.2.

($\Rightarrow$) Assume there is $v \in \mathrm{Hom}_{\mathsf{WK}}$ such that $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} >^{\wedge}_{\mathsf{WK}} \mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\}$. There are three cases to consider. First, $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = t$ and $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = f$. By Fact 2.3.1, $v \in \mathrm{Hom}_{\mathsf{B}_2}$ and therefore it is such that $\Gamma \nvDash_{\mathbf{CL}} \Delta$. Second, $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = t$ and $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = e$. By Fact 2.3.1, there is a propositional variable $p \in Var(\Delta)$ such that $v(p) = e$, hence $p \notin Var(\Gamma)$. Thus, $Var(\Delta) \nsubseteq Var(\Gamma)$. Third, $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = f$ and $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = e$. Again, by Fact 2.3.1, there is a variable $p \in Var(\Delta)$ and $p \notin Var(\Gamma)$. Thus, $Var(\Delta) \nsubseteq Var(\Gamma)$. So, either way, either $Var(\Delta) \nsubseteq Var(\Gamma)$ or $\Gamma \nvDash_{\mathbf{CL}} \Delta$.

($\Leftarrow$) Suppose that $\Gamma \nvDash_{\mathbf{CL}} \Delta$ or $Var(\Delta) \nsubseteq Var(\Gamma)$. In the first case, if $\Gamma \nvDash_{\mathbf{CL}} \Delta$, there is $v \in \mathrm{Hom}_{\mathsf{B}_2}$ such that for all $\gamma \in \Gamma$, $v(\gamma) = t$ while for all $\delta \in \Delta$, $v(\delta) = f$. But $v \in \mathrm{Hom}_{\mathsf{WK}}$ so it is such that $\mathtt{min}^{\wedge}\{v(\gamma)|\gamma \in \Gamma\} >^{\wedge}_{\mathsf{WK}} \mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\}$. In the second case, if $Var(\Delta) \nsubseteq Var(\Gamma)$, then there is some $p \in Var(\Delta)$ such that $p \notin Var(\Gamma)$. So it is possible to construct a valuation $v \in \mathrm{Hom}_{\mathsf{WK}}$ such that if $q \in Var(\Gamma)$ then $v(q) = t$, while $v(q) = e$ otherwise. By Fact 2.3.1, v is such that $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} \in \{t, f\}$ but $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = e$, so $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} >^{\vee}_{\mathsf{WK}} \mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\}$.

□

Theorem 3.4.2. $\Gamma \vDash_{\mathbf{CL}^{\mathrm{sl}}} \Delta$ *if and only if for every valuation* $v \in$ *Hom*$_{\mathsf{WK}}$: $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} \leq^{\vee}_{\mathsf{WK}} \mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\}$

Proof. Both directions are proved by contraposition.

($\Rightarrow$) Assume that there is $v \in \mathrm{Hom}_{\mathsf{WK}}$ such that $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} >^{\vee}_{\mathsf{WK}} \mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\}$. There are three cases to consider. First, $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = t$ and $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = f$. By Fact 2.3.1, $v \in \mathrm{Hom}_{\mathsf{B}_2}$, so it is also such that $\Gamma \nvDash_{\mathbf{CL}} \Delta$. Second, $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = e$ and $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = t$. By Fact 2.3.1, there is a variable $p \in Var(\Gamma)$ such that $v(p) = e$, hence $p \notin Var(\Delta)$. Thus, $Var(\Gamma) \nsubseteq Var(\Delta)$. Third, $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = e$ and $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} = f$. Again, by Fact 2.3.1, there is a variable $p \in Var(\Gamma)$ and $p \notin Var(\Delta)$. Hence, $Var(\Gamma) \nsubseteq Var(\Delta)$. So, either way, $Var(\Gamma) \nsubseteq Var(\Delta)$ or $\Gamma \nvDash_{\mathbf{CL}} \Delta$.

($\Leftarrow$) Suppose that $\Gamma \nvDash_{\mathbf{CL}} \Delta$ or $Var(\Gamma) \nsubseteq Var(\Delta)$. In the first case, if $\Gamma \nvDash_{\mathbf{CL}} \Delta$, then there is $v \in \mathrm{Hom}_{\mathsf{B}_2}$ such that for all $\gamma \in \Gamma$, $v(\gamma) = t$ while for all $\delta \in \Delta$, $v(\delta) = f$. But $v \in \mathrm{Hom}_{\mathsf{WK}}$ and it is such that $\mathtt{min}^{\wedge}\{v(\gamma)|\gamma \in \Gamma\} >^{\vee}_{\mathsf{WK}} \mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\}$. In the second case, if $Var(\Gamma) \nsubseteq Var(\Delta)$, then there is at least one $p \in Var(\Gamma)$ such that $p \notin Var(\Delta)$. So it is possible to construct a valuation $v \in \mathrm{Hom}_{\mathsf{WK}}$ such that $v(q) = f$ fro all $q \in Var(\Delta)$, while $v(q) = e$ otherwise. So by Fact 2.3.1, v is such that $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} = e$ and $\mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\} \in \{t, f\}$, so $\mathtt{min}^{\wedge}\{v(\gamma) \mid \gamma \in \Gamma\} >^{\vee}_{\mathsf{WK}} \mathtt{max}^{\vee}\{v(\delta) \mid \delta \in \Delta\}$.

□

For finite Γ and Δ, the former characterizations are equivalent to these:

$$\Gamma \vDash_{\mathbf{CL}^{\mathbf{sr}}} \Delta \text{ iff for all } v \in \mathrm{Hom}_{\mathsf{WK}} : v(\bigwedge_{\mathsf{WK}}\Gamma) \leq^{\wedge}_{\mathsf{WK}} v(\bigvee_{\mathsf{WK}}\Delta)$$

$$\Gamma \vDash_{\mathbf{CL}^{\mathbf{sl}}} \Delta \text{ iff for all } v \in \mathrm{Hom}_{\mathsf{WK}} : v(\bigwedge_{\mathsf{WK}}\Gamma) \leq^{\vee}_{\mathsf{WK}} v(\bigvee_{\mathsf{WK}}\Delta)$$

Similarly to $\mathbf{CL^{pr}}$ and $\mathbf{CL^{pl}}$, also $\mathbf{CL^{sr}}$ and $\mathbf{CL^{sl}}$ can thus be seen as consequence relations based on an order over WK—see Facts 3.1.3 and 3.1.2. However, in that case the minimum on the left and the maximum on the right where over the same order, hence in the pure right case, finite premises are conjoined according to the Sobociński conjunction, while in the pure left case, conclusions are disjoined according to the Sobociński disjunction. In the simple companions, instead, the order associated to the minimum on the left and the maximum on the right changes in order to coincide with the order of Weak Kleene conjunction and Weak Kleene disjunction, which, as it was discussed in section 2.2, induce different orders.

These characterization results provide an insight into these new systems which also allows to compare them with their variable inclusion siblings. For instance, as it was discussed in section 3.1, $\mathbf{CL^{pr}}$ and $\mathbf{CL^{pl}}$ coincide with the first degree fragment of **PAI** and **DD** respectively, if single premise/single conclusion inferences are considered. This correspondence hinges on the variable-inclusion proviso built into pure logics, which requires that neither side of a sequent be empty—–just as a conditional cannot lack either an antecedent or a consequent. However, one may argue that the systems which most appropriately stand in analogy with conditionals complying with Parry or dual-Parry's proscriptive principles, should not impose extra requirements to the proscription—such as non-emptiness—when this proviso is applied to the level of consequence, given that consequence, unlike conditionals, does not rule out in principle the possibility of empty premises or empty conclusions. For this reason, **PAI** stands to Parry's principle for conditionals as $\mathbf{CL^{sr}}$, rather than $\mathbf{CL^{pr}}$, stands to the same principle for inferences. Likewise, **DD** relates to the dual-Parry principle for conditionals just as $\mathbf{CL^{sl}}$, rather than $\mathbf{CL^{pl}}$ does to its inferential counterpart. This analogy deserves to be studied more extensively, generalizing this approach to other kinds of conditionals and the consequence relations.

To conclude this chapter, Figures 3.1 and 3.2 summarize the main features and distinctions among the systems discussed. The first diagram considers the multiple premise/multiple conclusion setting, while the second shows how some systems collapse when restricted to single premise/single conclusion inferences. As this last diagram shows, the simple variable inclusion companions of **CL** do not collapse with any other logic and they are the only systems which remain substructural since in this restricted setting, the universal and universal pure companions of **CL** coincide, respectively, with the WK logics and the pure companions, both closed under the properties listed in Definition 2.2.2.

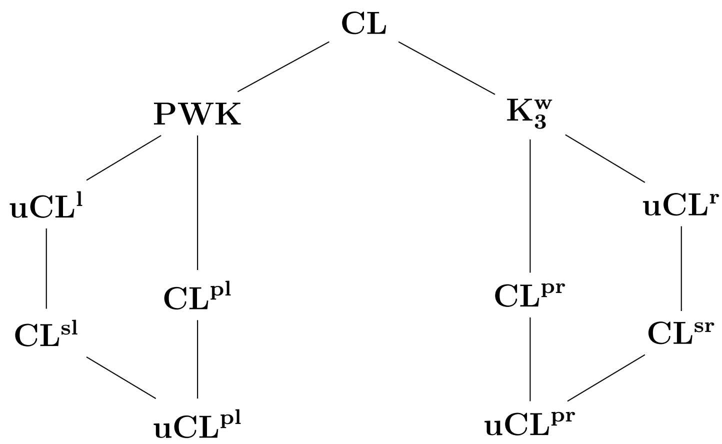

Figure 3.1 Comparing variable inclusion logics (multiple premises/conclusions)

To conclude, Tables 3.1 and 3.2 collect the key properties discussed above and offer a side-by-side comparison of the systems considered so far.

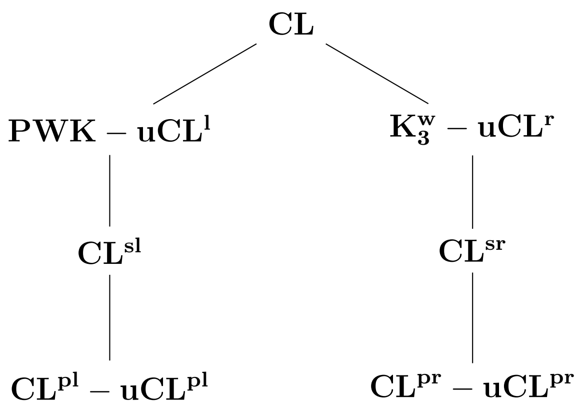

Figure 3.2 Comparing variable inclusion logics (single premise/conclusion)

CL	$\varphi \vDash \psi \vee \neg\psi$	$\varphi \wedge \neg\varphi \vDash \psi$	$\varphi \wedge \psi \vDash \psi$	$\varphi \vDash \varphi \vee \psi$	$\varphi, \psi \vDash \psi$	$\varphi \vDash \varphi, \psi$
K$_3^{\mathbf{w}}$	✗	✓	✓	✗	✓	✓
CL$^{\mathbf{pr}}$	✗	✗	✓	✗	✓	✓
uCL$^{\mathbf{r}}$	✗	✓	✓	✗	✓	✗
CL$^{\mathbf{sr}}$	✗	$\varphi \wedge \neg\varphi \vDash \varnothing$	✓	✗	✓	✗
uCL$^{\mathbf{pr}}$	✗	✗	✓	✗	✓	✗

Table 3.1: Comparisons among Right Companions of **CL**

CL	$\varphi \vDash \psi \vee \neg\psi$	$\varphi \wedge \neg\varphi \vDash \psi$	$\varphi \wedge \psi \vDash \psi$	$\varphi \vDash \varphi \vee \psi$	$\varphi, \psi \vDash \psi$	$\varphi \vDash \varphi, \psi$
PWK	✓	✗	✗	✓	✓	✓
CL$^{\mathbf{pl}}$	✗	✗	✗	✓	✓	✓
uCL$^{\mathbf{l}}$	✓	✗	✗	✓	✗	✓
CL$^{\mathbf{sl}}$	$\varnothing \vDash \psi \vee \neg\psi$	✗	✗	✓	✗	✓
uCL$^{\mathbf{pl}}$	✗	✗	✗	✓	✗	✓

Table 3.2: Comparisons among Left Companions of **CL**

Chapter 4

From Two-component Content to Two-address Semantics

The previous chapters presented containment logics as companions to other logics. These characterizations justify their interpretation as logics of content inclusion in the Yablovian sense, by making explicit that their consequence relation amounts to implication plus inclusion of topic captured by inclusion of variables.

However, there is another way to show that these systems are logics of content inclusion, though it eventually coincides with it. Like the former strategy, this option encodes Yablo's double-barreled analysis of content, though treating alethic and topical ascriptions independently and on a par, yielding a two-address semantic framework. This chapter develops this strategy and shows how it captures containment logics as logics of content inclusion. The discussion is confined to structural logics, leaving applications to substructural systems for future work.

Some of the material of this chapter draws on joint work with Damian Szmuc (SADAF–CONICET, University of Buenos Aires) which is part of an article (Szmuc and Zirattu, 2025.)

4.1 True, False, On-topic, Off-topic

Section 2.5 presented the interpretation offered by Beall, 2016 for the infectious value of $\mathbf{K_3^w}$ as the value to be assigned to statements which are off-topic with respect to a given subject matter. This interpretation was then extended to sets of statements in section 3.2 as a further motivation for universal companions. Though Beall's view specifically considered scientific theories as a way to justify the adoption of a $\mathbf{K_3^w}$ consequence relation for the closure of the theory itself, rather than the classical one, this interpretation is also applicable to the demand of relevance. That is, given some subject matter fixed by the premises of an argument, if the argument is to be analytic, then its conclusions should not go off-topic, while if it is to be ampliative, then conclusions could go off-topic but building on the premises' subject matter.

Beall's interpretation is generalized in Song et al., 2023, where the authors propose to assess a statement being off-topic or not independently of its truth or falsity, thereby placing alethic and topical statuses on an equal footing. So, for instance, although both "The Moon is made of green cheese" and "The Moon is a satellite" are off-topic with respect to Euclidean geometry, the former happens to be false, while the latter is true. As Song et al., 2023 show, whether this distinction for off-topic statements is relevant will depend on the choice of the preferred consequence relation, but postulating in principle that this is not the case closes off several legitimate possibilities.

These considerations motivate Song et al., 2023 to develop a semantic framework in which formulas are evaluated through parallel assignments—

one tracking truth or falsity, and the other determining whether they are on- or off-topic—similarly to the proposal advanced in Herzberger, 1973 and Woodruff, 1973, where instead the second coordinate ranges over meaningfulness and meaninglessness. More formally, suppose that truth and falsity—denoted by t and f, respectively—are exhaustive and exclusive and that the off-topic value—denoted by $\otimes$—is infectious over the on-topic value—denoted by $\odot$. Suppose further that the operators are topic-transparent. Under these assumptions, the alethic and topical assignments extend to the formulas of $\mathscr{L}$ as follows:

1a $\quad v_a(\neg\varphi) = t \iff v_a(\varphi) = f$

1b $\quad v_a(\varphi \wedge \psi) = t \iff v_a(\varphi) = t$ and $v_a(\psi) = t$

1c $\quad v_a(\varphi \vee \psi) = t \iff v_a(\varphi) = t$ or $v_a(\psi) = t$

2a $\quad v_t(\neg\varphi) = \odot \iff v_t(\varphi) = \odot$

2b $\quad v_t(\varphi \wedge \psi) = \odot \iff v_t(\varphi) = \odot$ and $v_t(\psi) = \odot$

2c $\quad v_t(\varphi \vee \psi) = \odot \iff v_t(\varphi) = \odot$ and $v_t(\psi) = \odot$

It is easy to check that these valuations correspond, respectively, to homomorphisms from $\mathsf{Fml}(\mathscr{L})$ to B_2 and IS_2—which, over a language $\mathscr{L}$ including both $\wedge$ and $\vee$, is such that $x \vee y \approx x \wedge y$ (see Figure 2.2). Consequently, the two assignments can be merged into a single two-address semantics, in which the underlying semantic structure is the direct product of an algebra of alethic values and an algebra of topics. In this case, the structure $\mathsf{B}_2 \times \mathsf{IS}_2$ (see Figure 4.1) represents the interaction between these two semantic dimensions when truth and falsity behave classically and all logical operators are topic transparent. A statement may thus receive one of the values $\langle t, \otimes \rangle$ or $\langle t, \odot \rangle$ if it is true and off-topic or on-topic, respectively, and $\langle f, \otimes \rangle$ or $\langle f, \odot \rangle$ if it is false and off-topic or on-topic. For convenience, such values will be written in concatenated form, omitting brackets and commas (e.g. $t\odot$).

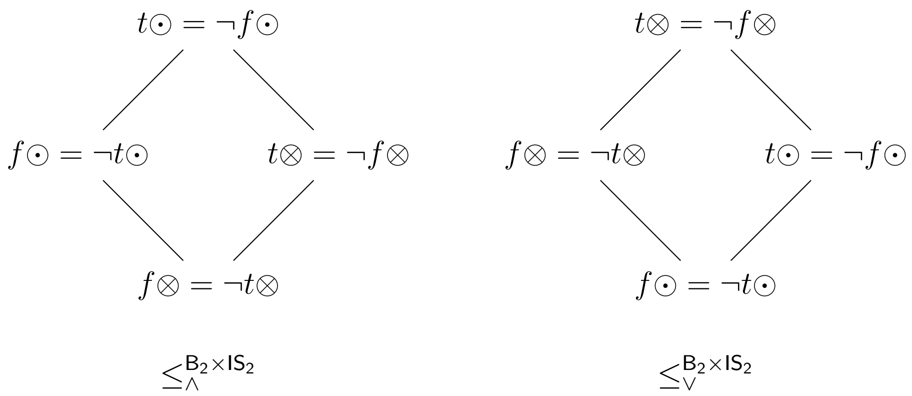

Figure 4.1

Remarkably, this approach is not tight to any specific choice for either semantic coordinate and, most importantly, the selections of the two structures to be combined are independent. So, for instance, if truth and falsity are taken to be exhaustive but not exclusive, or exclusive but not exhaustive, then this idea may be represented adding an extra alethic status, for gappy or glutty statements, respectively. The reification of one such status can then be implemented simply employing K_3 instead of B_2 in the first coordinate of the two-address semantics, or DM_4 if both statuses were permitted. It is worth anticipating though that within the two-address strategy, representing more (alethic or topical) statuses by adding more values in the respective algebraic structure, is not the only option to model this and analogous situations—as will be discussed in section 4.4. Moreover, it will be argued that different assumptions on the way topics are assigned can be represented with the employment of different structures on the other coordinate.

Before proceeding, it is important to emphasize that although Song et al., 2023 do not consider Yablo's two-component view of content, their proposal can naturally be read under this thesis. As discussed in section 1.4, the content of a declarative statement can be represented as a pair of two coordinates, the first representing its truth-conditions, the latter its subject-matter. This idea is naturally depicted through the two-address semantics described above. The two-address strategy is also employed in Berto, 2022, this time explicitly inspired by the Yablovian thesis, where parallel assignments also coordinate to determine the alethic value on the one hand and the topic of a statement on the other, though in this case more possibilities other than the on-off-topic range are included. However, this fine-grainedness is not relevant inasmuch as the topical structure remains of the same kind. For instance, if (as in both works) conjunction, disjunction and negation are taken to be topic transparent, then the topic of a statement is nothing but the sum of all the topics of its propositional components, hence such structure should be an involutive semilattice which, furthermore, satisfies $\neg x \approx x$. In this case, considering IS_2 already suffices to capture the relevant *logical* relations between topics—such as whether two statements are *necessarily* about the the same topic, or whether the topic of one statement is *always* included within that of another—given that it generates the subvariety of involutive semilattices satisfying transparency of negation (see section 2.2)[1].

With the motivation for two-address semantics as a formal model of two-

[1]This also emerges in the discussion of other relevance logics in Iacona, n.d. where the fine-grainedness of the topic assignments does not make a difference when considering valid conditionals only. Moreover, similar considerations can be seen as motivating the defense of Beall's interpretation in Carrara and Pra Baldi, n.d.

component content established, it is now useful to discuss how the topical coordinate can be adjusted to capture different ways the topic of a complex formula arises from its components—as done in Szmuc and Zirattu, 2025. In particular, just as relaxing the assumption that the negation of a true statement is always and only false may require structures such as K_3 and DM_4 to reify more alethic statuses, relaxing the assumption that each statement shares its topic with its negation— *i.e.* allowing negation to be topic-transformative— may lead to a similar modification of the topical coordinate. Recall that the employment of IS_2 is committed to the fact that for each element x, $x \approx \neg x$, while, dropping this assumption altogether, can be accounted for substituting IS_2 with the four-element involutive semilattice IS_4 (see Figure 2.2). In fact, IS_4 guarantees transparency of conjunction and disjunction as well as the fact that a statement and its double negation are about the same topic, but rejects in principle transparency of negation.

Although Song et al., 2023 do not consider this case, some semantics for $\mathbf{AC}_{\texttt{fde}}$ that have been proposed *e.g.* in Szmuc and Zirattu, 2025, Fine, 2016a and Randriamahazaka, 2022, correspond to this choice for the second coordinate of the two-address framework[2]. Hence, the elements of IS_4 can be naturally interpreted by extending the reading proposed in Song et al., 2023, taking into account the role of negation. As before, $\odot$ corresponds to statements that are on-topic and whose negation is also on-topic, while $\otimes$ represents off-topic statements whose negation is likewise off-topic.

[2]Notice though that neither Fine, 2016a nor Randriamahazaka, 2022 explicitly frame their proposals as in the present discussion. While Randriamahazaka, 2022 does not provide any explicit philosophical motivation for the proposed semantics—then addressed in Szmuc and Rubin, 2022 as a variation of the nonsense interpretation (see section 2.5)—Fine, 2016a frames it within the author's discussion of partial truth. In any case, the resulting structures are isomorphic to the two-address semantics for $\mathbf{AC}_{\texttt{fde}}$ discussed here.

The new elements, instead, *i.e.* $\oplus$ and $\ominus$, are assigned to statements for which negation changes the topic: $\oplus$ corresponds to statements that are on-topic but whose negation is off-topic, and $\ominus$ to off-topic statements whose negation is on-topic. Therefore, $\oplus$ corresponds to the negation of $\ominus$ and vice-versa, while $\odot$ and $\otimes$ are fixed-points of negation.

In summary, since the choices for the two coordinates in the two-address semantics are independent, assuming that logical operators are topic-transparent determines IS_2 as the topical structure, whereas allowing negation to change the subject matter of a statement can be captured through IS_4. Either way, the alethic coordinate can be B_2 if no truth gaps or gluts are admitted, K_3 if one is allowed, and DM_4 if both are permitted.

At this point, it is worth pausing to recall what was mentioned in section 1.3. It was argued that although in principle there seems to be no definitive reason to favor the two-component over the one-component account of content for declarative statements—namely separating truth from topic or not—the main virtue of the former, corresponding to the Yablovian view, is its neutrality. The two-address framework can be seen as an example of this feature, given that it is flexible enough to accommodate classical alethic assumptions without necessarily leading to potentially undesired outcomes such as the synonymy of logical impossibilities—as they may well be equivalent in their first coordinate but not necessarily on the other. Moreover, this framework is general enough to allow for the implementation of distinct ideas regarding how the algebra of topics should look like, from the more standard topic-transparent case to more refined ones. Indeed, although the discussion so far has focused on the case in which negation is topic-transformative yet still involutive, this merely lays the groundwork for more flexible settings—*e.g.* as those suggested at the end of section 4.4. Another advantage of the two-address strategy—–already noted in Song et al., 2023 as well as Szmuc and Zirattu, 2025 and further examined in this and the next section—–lies in its ability to provide a unified framework for a range of logics that can be distinguished by whether their notion of logical consequence can be defined as topic-neutral or topic-sensitive.

There is, however, at least one aspect of the two-address strategy which may be criticized in terms of lack of generality, as discussed in Paoli et al., 2026: all variants of this framework share the assumption that the same range of alethic statuses is available for every topic. In fact, the direct product construction yields that if truth is assumed to be, *e.g.*, classical for some topic, then it will be so for any topic. Nevertheless, it is *prima facie* plausible that when one topic subsumes another, the former may admit a richer range of alethic statuses than the latter, insofar as it encompasses a broader class of propositions, and there is no obvious reason to assume that such an expansion must be non-proper. One possible response is to constrain the alethic statuses of atomic topics through suitable valuations of atomic formulas. This can be achieved by adjusting the range of the alethic coordinate in the direct product construction so that subtopics are assigned fewer alethic statuses. This adjustment may solve this specific point though it may be unsatisfactory for its *ad hoc* flavor. Moreover, it can be argued that the two-address framework represents a particular instance of a more general strategy to modeling the interaction between truth and subject matter. One such strategy is the Płonka construction (employed, *e.g.*, Bonzio et al., 2022, Randriamahazaka, 2024), which consists in the construction of an algebra from a family of component algebras, called fibers, arranged over a topical structure, where each fiber can potentially be of any kind[3]. In this sense, the direct product of the alethic and topical algebras is just a Płonka construction where all fibers are isomorphic. However, it is worth noting that, while generality is often an advantage, it should be appropriately constrained. Namely, just as there may be reasons to admit, for instance, additional alethic statuses for larger topics, it is not as intuitive to justify why the opposite or even more arbitrary combinations should be allowed, as permitted by the Płonka framework. This issue raises unresolved questions and asks for a more thorough examination. For now, though, it will be set aside in order to proceed with the discussion.

4.2 Parallel Preservation of Truth and Topic

The two-address strategy outlined above provides a semantic framework for a wide range of logics, including, crucially, several logics of content inclusion. In order to show how this is possible, it is useful to recall what content inclusion amounts to. On the one hand, there has to be an implication—*e.g.* as left-to-right truth preservation or right-to-left falsity preservation—and, on the other hand, inclusion of topic in one direction or the other, depending on the side of content inclusion to be captured. This characterization has a clear separation of the alethic side from the topical one, which very naturally coincides with the parallel treatment thereof within the two-address construction. Whereas in the previous chapters these features were captured by entailment in a logic equipped with a syntactic filter

[3]With the restriction that the fibers are all algebras of the same language and that this language contains at least an operation symbol of arity greater than or equal to 2.

of variable inclusion, the present framework places the two aspects on an equal footing, thereby allowing the definition of a notion of logical consequence that coincides with preservation of the relevant alethic property, the preservation of some topical status, or both simultaneously.

To make this precise, consider the simplest case: classical truth and falsity and topic-transparent operators. As argued above, this setting can be adequately represented by the direct product between B_2 and IS_2. Logical consequence can then be defined by means of matrices, by selecting an appropriate set of designated values from among the available ones, namely $t\odot$, $t\otimes$, $f\odot$, and $f\otimes$. Several choices are available, depending on which notion of content inclusion one is to capture. Consider the *analytic* version of such proviso as preservation of truth and topic from premises to conclusions. The first alone can be captured by designating the values having t as their first coordinate, namely $t\odot$ and $t\otimes$, while for the second, which in the on-off topic setting amounts to preservation of on-topicality, the values with $\odot$ as their second coordinate should be designated, hence $t\odot$ and $f\odot$. The convergence of these two aspects which should model content inclusion from right to left, may be represented in two ways that are in principle distinct and—as will be shown—coincide only in certain circumstances. One such way is to consider the *intersection* of such sets of designated values, hence preserving on-topic truth, which in this case is only $t\odot$—call this the *simultaneous* option. The other way is, instead, to employ a bundle of matrices where each one takes care of one status to be preserved at a time, hence preserving truth *and* on-topicality in parallel—call it the *parallel* option.

Now, consider the other direction of content inclusion as preservation of falsity and on-topicality from conclusions to premises. As before, these two aspects can be separately captured, respectively, by preserving $\{f\odot, f\otimes\}$ from right to left, hence designating $\{t\odot, t\otimes\}$, and by preserving $\{t\odot, f\odot\}$ also from right to left, thus designating $\{t\otimes, f\otimes\}$. As in the previous case, one option is to consider the resulting matrix bundle—the *parallel* option. Another option is intersecting the requirement of right-to-left falsity and on-topicality preservation, that is, preserving only $f\odot$ from conclusions to premises, and therefore considering only one matrix with $\{t\odot, t\otimes, f\otimes\}$ as its set of designated values—the *simultaneous* way. Notice that this strategy is dual to the analogous one outlined above for right-to-left content inclusion. In the former case, one option was to intersect the sets of *designated* values, while in this case the intersection is performed between the sets of *undesignated* values. In this way, the resulting set of designated values is given by the *union* of the original ones. This modification is needed to capture the correct side of topic preservation. If also in this case one considered the intersection of the sets of designated values, the resulting consequence relation would be preserving only $t\otimes$, meaning that from conclusions to premises also some off-topic value, namely $f\otimes$, would be preserved, violating the topical condition.

The above considerations apply also to cases in which truth and falsity are not exhaustive and exclusive and to treatments of negation as topic-transformative. In the former case, for instance, if one admits both truth gaps and gluts, which correspond to the elements n and b in $\mathsf{DM_4}$, then plain truth preservation in that context will amount to preservation of all values with either t or b on their left coordinate. In the latter case, preservation of on-topicality not only requires designating the values with $\odot$ on the right coordinate, but also $\oplus$, while reverse-topic preservation amounts to designating the values with $\otimes$ or $\ominus$ on the topical side. As before, content inclusion can be captured either appropriately intersecting the original sets of designated values or undesignated values, or combining in a bundle the matrices capturing in parallel the alethic and topical preservation.

In light of the preceding remarks, the following results show how the two-address framework provides semantics for the structural logics considered so far, including both regular and balanced cases for weak and pure companions of a given system. This section illustrates the parallel strategy while the next focuses on the simultaneous one.

First, it is useful to characterize preservation of on-topicality in one direction or the other both in the regular and balanced case. To do so, consider the topical structures in isolation, starting with IS_2. First, notice that $\otimes$ behaves infectiously as mentioned earlier:

Fact 4.2.1. For any $v \in \mathrm{Hom}_{\mathsf{IS}_2}$ and $\varphi \in Fml(\mathscr{L})$, $v(\varphi) = \otimes$ if and only if there is some $p \in Var(\varphi)$ such that $v(p) = \otimes$.

Proof. The right-to-left direction can be immediately established by inspection of the operations over IS_2, the other direction can be proved by contraposition. Assume that for all $p \in Var(\varphi)$, $v(p) = \odot$, once more by the definition of the operators, $v(\varphi) = \odot$. □

With this result, it is possible to characterize left-to-right preservation of on-topicality over IS_2[4].

Theorem 4.2.1. *Let* $\mathfrak{M}_\odot$ *be the matrix* $\langle \mathsf{IS}_2, \{\odot\} \rangle$*. For all* $\Gamma, \Delta \subseteq Fml(\mathscr{L})$,

[4]When restricted to single-conclusion inferences, Theorems 4.2.1, 4.2.2 and 4.2.6 displayed below, follow from Theorems 4.1 and 4.4 in Paoli et al., 2021.

$\Gamma \vDash_{\mathfrak{M}_{\odot}} \Delta$ *if and only if* $\exists \Delta' | \Delta' \neq \varnothing$ *and* $\Delta' \subseteq \Delta$ *such that* $Var(\Delta') \subseteq Var(\Gamma)$.

Proof. Both directions are proved by contraposition.

(⇒) Suppose that there is no $\Delta' | \Delta' \neq \varnothing$ and $\Delta' \subseteq \Delta$ such that $Var(\Delta') \subseteq Var(\Gamma)$. Thus each such Δ', and in particular each $\delta \in \Delta$, has some variable p such that $p \notin Var(\Gamma)$. Then construct a valuation $v \in \mathrm{Hom}_{\mathsf{IS}_2}$ such that $v(q) = \odot$ if $q \in Var(\Gamma)$, $v(q) = \otimes$ otherwise. By the first assumption and Fact 4.2.1, $v(\gamma) = \odot$ for all $\Gamma \in \Gamma$ while $v(\delta) = \otimes$ for all $\delta \in \Delta$. Therefore $\Gamma \nvDash_{\mathfrak{M}_{\odot}} \Delta$.

(⇐) Suppose that $\Gamma \nvDash_{\mathfrak{M}_{\odot}} \Delta$. Hence there is some $v \in \mathrm{Hom}_{\mathsf{IS}_2}$ such that $v(\gamma) = \odot$ for all $\gamma \in \Gamma$ but $v(\delta) = \otimes$ for all $\delta \in \Delta$. By Fact 4.2.1, for each $\delta \in \Delta$ there is some $p \in Var(\delta)$ such that $v(p) = \otimes$, but by the same fact there is no $q \in Var(\Gamma)$ such that $v(q) = \otimes$, meaning that there is no $\delta \in \Delta$ such that $Var(\delta) \subseteq Var(\Gamma)$, therefore for all $\Delta' | \Delta' \neq \varnothing$ such that $\Delta' \subseteq \Delta$, $Var(\Delta') \nsubseteq Var(\Gamma)$.

□

The other direction of on-topicality preservation needed for left-to-right content inclusion, as expected, is captured by designating $\otimes$ (the proof is analogous to the previous).

Theorem 4.2.2. *Let* $\mathfrak{M}_{\otimes}$ *be the matrix* $\langle \mathsf{IS}_2, \{\otimes\} \rangle$. *For all* $\Gamma, \Delta \subseteq Fml(\mathscr{L})$, $\Gamma \vDash_{\mathfrak{M}_{\otimes}} \Delta$ *if and only if* $\exists \Gamma' | \Gamma' \neq \varnothing$ *and* $\Gamma' \subseteq \Gamma$ *such that* $Var(\Gamma') \subseteq Var(\Delta)$.

For the balanced case, the following fact serves to characterize on-topicality preservation.

Fact 4.2.2. For any $v \in \mathrm{Hom}_{\mathsf{IS}_4}$ and $\varphi \in Fml(\mathscr{L})$:

(i) $v(\varphi) = \otimes$ if and only if either there is some $p \in Var(\varphi)$ such that $v(p) = \otimes$, or there are both some $q \in Var^{+}(\varphi)$ and some $r \in Var^{-}(\varphi)$ such that $v(q), v(r) \in \{\oplus, \ominus\}$ and $v(q) = v(r)$, or there are some q, r with the same polarity such that $v(q), v(r) \in \{\oplus, \ominus\}$ and $v(q) \neq v(r)$.

(ii) $v(\varphi) = \oplus$ if and only if for all $p \in Var^{+}(\varphi)$, $v(p) \in \{\oplus, \odot\}$ and for all $q \in Var^{-}(\varphi)$, $v(q) \in \{\ominus, \odot\}$ and moreover there is at least some $r \in Var(\varphi)$ such that $v(r) \neq \odot$.

(iii) $v(\varphi) = \ominus$ if and only if for all $p \in Var^{+}(\varphi)$, $v(p) \in \{\ominus, \odot\}$ and for all $q \in Var^{-}(\varphi)$, $v(q) \in \{\oplus, \odot\}$ and moreover there is at least some $r \in Var(\varphi)$ such that $v(r) \neq \odot$.

(iv) $v(\varphi) = \odot$ if and only if for all $p \in Var(\varphi)$, $v(p) = \odot$

Proof. All items can be proved noting that, by the properties of involutive semilattices described in section 2.2 (*i.e.* for all x, y, $x \approx \neg\neg x$, $x \vee y \approx x \wedge y$, $\neg(x \vee y) \approx \neg x \vee \neg y$), the following equivalence holds for any $\varphi \in Fml(\mathscr{L})$ and any $v \in \mathrm{Hom}_{\mathsf{IS}_4}$:

$$v(\varphi) = \left(\bigvee_{p \in Var^{+}(\varphi)} v(p) \right) \vee \left(\bigvee_{q \in Var^{-}(\varphi)} \neg v(q) \right)$$

With this and the way operators over IS_4 are defined, the right-to-left direction of each point follows immediately. The other direction can be proved by contraposition. For instance, for point (i) negating all disjuncts leads to three possible scenarios each corresponding to the right-hand side of points (ii)-(iv), implying that $v(\varphi) \neq \otimes$. A similar reasoning applies to the other items.

□

Thus, left-to-right preservation of topic when negation may be topic-transformative can be represented over IS_4 as follows.

Theorem 4.2.3. *Let $\mathfrak{M}_{\odot\oplus}$ be the matrix $\langle \mathsf{IS}_4, \{\odot, \oplus\}\rangle$. For all $\Gamma, \Delta \subseteq Fml(\mathscr{L})$, $\Gamma \vDash_{\mathfrak{M}_{\odot\oplus}} \Delta$ if and only if $\exists \Delta' | \Delta' \neq \varnothing$ and $\Delta' \subseteq \Delta$ such that both $Var^+(\Delta') \subseteq Var^+(\Gamma)$ and $Var^-(\Delta') \subseteq Var^-(\Gamma)$.*

Proof. Both directions are proved by contraposition.

($\Rightarrow$) Suppose that there is no $\Delta' | \Delta' \neq \varnothing$ and $\Delta' \subseteq \Delta$ such that both $Var^+(\Delta') \subseteq Var^+(\Gamma)$ and $Var^-(\Delta') \subseteq Var^-(\Gamma)$. Thus each such Δ', and in particular each $\delta \in \Delta$, has some positive variable p such that $p \notin Var^+(\Gamma)$ or some negative variable q such that $q \notin Var^-(\Gamma)$. Then construct a valuation $v \in \mathrm{Hom}_{\mathsf{IS}_4}$ as follows:

$$v(r) = \begin{cases} \ominus & \text{if } r \in Var^+(\Delta) \setminus Var^+(\Gamma) \text{ and } r \notin Var^-(\Delta) \setminus Var^-(\Gamma), \\ \oplus & \text{if } r \in Var^-(\Delta) \setminus Var^-(\Gamma) \text{ and } r \notin Var^+(\Delta) \setminus Var^+(\Gamma), \\ \otimes & \text{if } r \in Var^+(\Delta) \setminus Var^+(\Gamma) \text{ and } r \in Var^-(\Delta) \setminus Var^-(\Gamma), \\ \odot & \text{otherwise.} \end{cases}$$

By this construction and the first assumption, in Γ no variable is assigned $\otimes$ and all variables appearing only positively receive value $\oplus$ while those appearing only negatively receive $\ominus$. Instead, in Δ all variables occurring only positively receive value $\ominus$ while those occurring only negatively are assigned $\oplus$ and those occurring in both ways get value $\otimes$. Thus, by Fact 4.2.2, $v(\gamma) \in \{\odot, \oplus\}$ for all $\gamma \in \gamma$ while $v(\delta) \in \{\otimes, \ominus\}$ for all $\delta \in \Delta$. Therefore $\Gamma \nvDash_{\mathfrak{M}_{\odot\oplus}} \Delta$.

($\Leftarrow$) Suppose that $\Gamma \nvDash_{\mathfrak{M}_{\odot\oplus}} \Delta$. Hence there is a valuation $v \in \mathrm{Hom}_{\mathsf{IS}_4}$ such that $v(\gamma) \in \{\odot, \oplus\}$ for all $\gamma \in \Gamma$ but $v(\delta) \in \{\otimes, \ominus\}$ for all $\delta \in \Delta$. Therefore, by Fact 4.2.2, for each $\delta \in \Delta$ there are two possibilities. The first possibility is that there is some $p \in Var(\delta)$ such that $v(p) = \otimes$, or there are both some $q \in Var^+(\delta)$ and some $r \in Var^-(\delta)$ such that $v(q), v(r) \in \{\oplus, \ominus\}$ and $v(q) = v(r)$, or there are some q, r with the same polarity such that $v(q), v(r) \in \{\oplus, \ominus\}$ and $v(q) \neq v(r)$. The second possibility is that for all $p \in Var^+(\delta)$, $v(p) \in \{\ominus, \odot\}$ and for all $q \in Var^-(\delta)$, $v(q) \in \{\oplus, \odot\}$ and moreover there is at least some $r \in Var(\delta)$ such that $v(r) \neq \odot$. But neither of these possibilities can hold for the formulas in Γ, therefore either way there is no non-empty $\Delta' \subseteq \Delta$ for which both $Var^+(\Delta') \subseteq Var^+(\Gamma)$ and $Var^-(\Delta') \subseteq Var^-(\Gamma)$.

□

A similar proof runs for the characterization of on-topicality preservation from conclusions to premises, which amounts to left-to-right off-topicality preservation.

Theorem 4.2.4. *Let $\mathfrak{M}_{\otimes\ominus}$ be the matrix $\langle \mathsf{IS}_4, \{\otimes, \ominus\}\rangle$. For all $\Gamma, \Delta \subseteq Fml(\mathscr{L})$, $\Gamma \vDash_{\mathfrak{M}_{\otimes\ominus}} \Delta$ if and only if $\exists \Gamma' | \Gamma' \neq \varnothing$ and $\Gamma' \subseteq \Gamma$ such that both $Var^+(\Gamma') \subseteq Var^+(\Delta)$ and $Var^-(\Gamma') \subseteq Var^-(\Delta)$.*

Notice that by the results proved so far the following correspondences follow—recall the definitions of (balanced)m and n-infectious extensions of a matrix (Definitions 3.1.5 and 3.1.6) and the characterization for such constructions (Theorems 3.1.1 and 3.1.2 and Theorems 3.1.5 and 3.1.6, respectively):

Theorem 4.2.5. *For all* $\Gamma, \Delta \subseteq Fml(\mathscr{L})$*:*

(i) $\Gamma \models_{\mathfrak{M}_{\odot}} \Delta$ *if and only if* $\Gamma \models_{\mathfrak{M}^{m}[e]} \Delta$

(ii) $\Gamma \models_{\mathfrak{M}^{\otimes}} \Delta$ *if and only if* $\Gamma \models_{\mathfrak{M}^{n}[e]} \Delta$

(iii) $\Gamma \models_{\mathfrak{M}^{\odot\oplus}} \Delta$ *if and only if* $\Gamma \models_{\mathfrak{M}^{m}[\pm]} \Delta$

(iv) $\Gamma \models_{\mathfrak{M}^{\otimes\ominus}} \Delta$ *if and only if* $\Gamma \models_{\mathfrak{M}^{n}[\pm]} \Delta$.

Given the above characterizations of topic preservation, the parallel combination with alethic preservation can be explicitly formalized through the following definitions[5].

Definition 4.2.1. Let $\mathfrak{M} = \langle \mathsf{A}, D\rangle$ be a matrix. Then $\boldsymbol{\mathfrak{M}}^{p\odot} = \{\mathfrak{M}_a, \mathfrak{M}_t\}$ is its *parallel topic preserving companion*, with $\mathfrak{M}_a = \langle \mathsf{A} \times \mathsf{IS}_2, D \times \{\odot, \otimes\}\rangle$ and $\mathfrak{M}_t = \langle \mathsf{A} \times \mathsf{IS}_2, A \times \{\odot\}\rangle$, while $\boldsymbol{\mathfrak{M}}^{pr\odot} = \{\mathfrak{M}_a, \mathfrak{M}_{rt}\}$ is its *parallel reverse-topic preserving companion*, with $\mathfrak{M}_{rt} = \langle \mathsf{A} \times \mathsf{IS}_2, A \times \{\otimes\}\rangle$. Similarly, $\boldsymbol{\mathfrak{M}}^{p\odot\oplus} = \{\mathfrak{M}_{ab}, \mathfrak{M}_{tb}\}$ is its *balanced parallel topic preserving companion*, with $\mathfrak{M}_{ab} = \langle \mathsf{A} \times \mathsf{IS}_4, D \times \{\odot, \otimes, \oplus, \ominus\}\rangle$ and $\mathfrak{M}_{tb} = \langle \mathsf{A} \times \mathsf{IS}_4, A \times \{\odot, \oplus\}\rangle$, while $\boldsymbol{\mathfrak{M}}^{pr\odot\oplus} = \{\mathfrak{M}_{ab}, \mathfrak{M}_{rtb}\}$ is its *balanced parallel reverse-topic preserving companion*, with $\mathfrak{M}_{rtb} = \langle \mathsf{A} \times \mathsf{IS}_4, A \times \{\otimes, \ominus\}\rangle$.

With this, the characterization of content inclusion in the parallel strategy can be proved (recall the definitions of (balanced) pure infectious extensions of a matrix from section 3.1).

[5]Notice though that for the "parallel" option it is not necessary to consider the direct product construction—as one may just employ matrices with a different algebraic reduct.

Theorem 4.2.6. *For all* $\Gamma, \Delta \subseteq Fml(\mathscr{L})$*:*

(i) $\Gamma \vDash_{\mathfrak{M}_{p\odot}} \Delta$ *if and only if* $\Gamma \vDash_{\mathfrak{M}^{pr}} \Delta$

(ii) $\Gamma \vDash_{\mathfrak{M}^{pr\odot}} \Delta$ *if and only if* $\Gamma \vDash_{\mathfrak{M}^{pl}} \Delta$

(iii) $\Gamma \vDash_{\mathfrak{M}^{p\odot\oplus}} \Delta$ *if and only if* $\Gamma \vDash_{\mathfrak{M}^{pr\pm}} \Delta$

(iv) $\Gamma \vDash_{\mathfrak{M}^{pr\odot\oplus}} \Delta$ *if and only if* $\Gamma \vDash_{\mathfrak{M}^{pl\pm}} \Delta$.

Proof. For each point notice that the alethic matrix ($\mathfrak{M}_a$ or $\mathfrak{M}_{ab}$) validates exactly the same inferences as the original $\mathfrak{M}$. In fact, the designated values of the former coincide on the left coordinate with those of the latter, while the right coordinate ranges over all elements of the involutive semilattice. Therefore, counterexamples over one matrix are straightforwardly translatable to the other and vice-versa. For the analogous reason, each topical matrix $\mathfrak{M}_t$, $\mathfrak{M}_{rt}$, $\mathfrak{M}_{tb}$ and $\mathfrak{M}_{rtb}$, validates exactly the same inferences as, respectively, $\mathfrak{M}_{\odot}$, $\mathfrak{M}_{\otimes}$, $\mathfrak{M}_{\odot\oplus}$ and $\mathfrak{M}_{\otimes\ominus}$[6]. Consequently, by Theorems 4.2.1, 4.2.2, 4.2.3 and 4.2.4, the parallel companion listed in each point validates the same inferences as $\mathfrak{M}$ that respect the variable inclusion filter imposed by the respective topical matrix, just in the same way as the pure (balanced) right-left companions of the same matrix do (recall Theorems 3.1.4, 3.1.3, 3.1.8 and 3.1.7, respectively). □

The proof of the above fact shows in what sense exactly the parallel strategy models content inclusion, namely capturing the pure (balanced) right or left variable inclusion companion of any (extending the result to

[6]One may wonder why, then, using direct product structures for this characterization in the first place. Of course one could replace $\boldsymbol{\mathfrak{M}_{p\odot}}$ with $\{\mathfrak{M}, \mathfrak{M}_{\odot}\}$, $\boldsymbol{\mathfrak{M}_{pr\odot}}$ with $\{\mathfrak{M}, \mathfrak{M}_{\otimes}\}$, $\boldsymbol{\mathfrak{M}_{p\odot\oplus}}$ with $\{\mathfrak{M}, \mathfrak{M}_{\odot\oplus}\}$ and $\boldsymbol{\mathfrak{M}_{pr\odot\oplus}}$ with $\{\mathfrak{M}, \mathfrak{M}_{\otimes\ominus}\}$. However, the comparison with Theorem 4.3.3 is simpler keeping the same algebraic construction, hence the two-address interpretation of formulas.

classes of matrices) given structural logic. In particular, the alethic matrix of the bundles above coincides with preservation of the relevant alethic property, which itself corresponds to derivability in some logic, while the topical matrix provides the appropriate syntactic filter of variable inclusion.

4.3 Simultaneous Preservation of Truth and Topic

Moving to the simultaneous strategy discussed above, the following definitions serve as an explicit formalization of the same.

Definition 4.3.1. Let $\mathfrak{M} = \langle \mathsf{A}, D \rangle$ be a matrix. Then $\mathfrak{M}_{s\odot} = \langle \mathsf{A} \times \mathsf{IS}_2, D \times \{\odot\} \rangle$ is its *simultaneous topic preserving companion* while $\mathfrak{M}_{sr\odot} = \langle \mathsf{A} \times \mathsf{IS}_2, A \setminus (\overline{D} \times \{\odot\}) \rangle$ is its *simultaneous reverse-topic preserving companion.* Similarly, $\mathfrak{M}_{s\odot\oplus} = \langle \mathsf{A} \times \mathsf{IS}_4, D \times \{\odot, \oplus\} \rangle$ is its *balanced simultaneous topic preserving companion* while $\mathfrak{M}_{sr\odot\oplus} = \langle \mathsf{A} \times \mathsf{IS}_4, (A \times \{\oplus, \ominus, \odot, \otimes\}) \setminus (\overline{D} \times \{\odot, \oplus\}) \rangle$ is its *balanced simultaneous reverse-topic preserving companion.*

These *companions* can be put in correspondence with constructions that were already presented in section 2.4, themselves employed to characterize the weak regular or balanced companions of a given logic. Before outlining this correspondence, it is worth pausing to provide an explicit proof of the following Theorem, anticipated in section 2.4.

Theorem 4.3.1.

$$\Gamma \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta \textit{ iff } \Gamma \vDash_{\mathfrak{M}} \Delta \textit{ and } \begin{cases} \textit{(i) } \varnothing \vDash_{\mathfrak{M}} \Delta \textit{ or} \\ \textit{(ii) } \exists \Gamma' \mid \Gamma' \subseteq \Gamma \textit{ and } \Gamma' \neq \varnothing \textit{ such that } \Gamma' \vDash_{\mathfrak{M}} \Delta \\ \quad \textit{and } Var^{+}(\Gamma') \subseteq Var^{+}(\Delta),\ Var^{-}(\Gamma') \subseteq Var^{-}(\Delta) \end{cases}$$

Proof. For simplicity, $\mathfrak{M}$ is taken to be a singleton but the proof can be straightforwardly extended to the more general case.

($\Rightarrow$) Suppose that $\Gamma \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$. There are two possibilities: either $\varnothing \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$ or $\varnothing \nvDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$.

- In the first case, there is no valuation $v \in \mathrm{Hom}_{\mathfrak{M}_{sr\odot\oplus}}$ such that $v(\delta) \in \overline{D} \times \{\odot, \oplus\}$ for all $\delta \in \Delta$, which in turn guarantees that there is no $v \in \mathrm{Hom}_{\mathfrak{M}_{sr\odot\oplus}}$ such that $v(\delta) \in \overline{D} \times \{\otimes, \ominus\}$ for all $\delta \in \Delta$. In fact, the existence of the latter would allow to construct a valuation behaving in the same way for the left coordinate but only assigning $\odot$ to the right, thus obtaining that all $\delta \in \Delta$ receive a value in $\overline{D} \times \{\odot\}$, contradicting the assumption.

  This means that all valuations over $\mathfrak{M}_{sr\odot\oplus}$ assign a value in $D \times \{\odot, \otimes, \oplus, \ominus\}$ for some $\delta \in \Delta$, hence implying that all valuations over $\mathfrak{M}$ assign a value in D for some $\delta \in \Delta$, thus that $\varnothing \vDash_{\mathfrak{M}} \Delta$. By monotonicity this also implies that $\Gamma \vDash_{\mathfrak{M}} \Delta$.

- In the second case, there is some $v' \in \mathrm{Hom}_{\mathfrak{M}_{sr\odot\oplus}}$ such that $v'(\delta) \in \overline{D} \times \{\odot, \oplus\}$ for all $\delta \in \Delta$. The fact that $\Gamma \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$, implies that there is some non-empty $\Gamma' \subseteq \Gamma$ such that $Var^+(\Gamma') \subseteq Var^+(\Delta)$ and $Var^-(\Gamma') \subseteq Var^-(\Delta)$. To see this, suppose that there is no such Γ', hence that for all $\gamma \in \Gamma$, $Var^+(\gamma) \nsubseteq Var^+(\Delta)$ or $Var^-(\gamma) \nsubseteq Var^-(\Delta)$. Then, by Fact 4.2.2 and the assumption that there is $v' \in \mathrm{Hom}_{\mathfrak{M}_{sr\odot\oplus}}$ such that $v'(\delta) \in \overline{D} \times \{\odot, \oplus\}$ for all $\delta \in \Delta$. it is possible to construct $v'' \in \mathrm{Hom}_{\mathfrak{M}_{sr\odot\oplus}}$ such that $v''(\gamma) \in A \times \{\otimes, \ominus\}$ for all $\gamma \in \Gamma$ while $v''(\delta) \in \overline{D} \times \{\odot, \oplus\}$, contradicting the initial assumption that $\Gamma \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$.

With this, it remains to show that $\Gamma' \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$. Call $\mathbf{\Gamma'}$ the set of all non-empty $\Gamma' \subseteq \Gamma$ such that $Var^+(\Gamma') \subseteq Var^+(\Delta)$ and $Var^-(\Gamma') \subseteq Var^-(\Delta)$, which by the former considerations is non-empty. Now suppose that any $\Gamma' \in \mathbf{\Gamma'}$ is such that $\Gamma' \nvDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$. Thus, $\bigcup_{\Gamma' \in \mathbf{\Gamma'}} \Gamma' \nvDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$. Therefore, there is some $v \in \mathrm{Hom}_{\mathfrak{M}_{sr\odot\oplus}}$ such that $v(\gamma) \notin \overline{D} \times \{\odot, \oplus\}$ for all $\gamma \in \Gamma'$ of each $\Gamma' \in \mathbf{\Gamma'}$ while $v(\delta) \in \overline{D} \times \{\odot, \oplus\}$ for all $\delta \in \Delta$. Moreover, given the construction of $\mathbf{\Gamma'}$ and Fact 4.2.2, $v(\gamma) \notin A \times \{\otimes, \ominus\}$ for all $\gamma \in \Gamma'$ of each $\Gamma' \in \mathbf{\Gamma'}$, *i.e.* $v(\gamma) \in D \times \{\odot, \oplus\}$ for all $\gamma \in \Gamma'$ of each $\Gamma' \in \mathbf{\Gamma'}$. Now construct a valuation $v^* \in \mathrm{Hom}_{\mathfrak{M}_{sr\odot\oplus}}$ such that it is the same as v for the left coordinate and such that it assigns on the right $\oplus$ to all $p \in Var^+(\Delta) \setminus Var^-(\Delta)$, $\ominus$ to all $p \in Var^-(\Delta) \setminus Var^+(\Delta)$, $\otimes$ to all $p \notin Var(\Delta)$ and $\odot$ otherwise. Thus, by Fact 4.2.2, v^* is such that $v^*(\varphi) \in D \times \{\odot, \oplus\}$ if $\varphi \in \bigcup_{\Gamma' \in \mathbf{\Gamma'}} \Gamma'$, $v^*(\varphi) = \overline{D} \times \{\odot, \oplus\}$ if $\varphi \in \Delta$ and $v^*(\varphi) \in A \times \{\ominus, \otimes\}$ otherwise. This implies that $v^*(\gamma) \notin \overline{D} \times \{\odot, \oplus\}$ for all $\gamma \in \Gamma$ while $v^*(\delta) \in \overline{D} \times \{\odot, \oplus\}$ for all $\delta \in \Delta$. But this would entail that $\Gamma \nvDash_{\mathfrak{M}_{\odot,\oplus}} \Delta$, contradicting the initial assumption.

Therefore, there has to be some $\Gamma' \in \mathbf{\Gamma'}$ such that $\Gamma' \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$. But this also implies that $\Gamma' \vDash_{\mathfrak{M}} \Delta$, since the existence of a counterexample over $\mathfrak{M}$ guarantees the existence of one over $\mathfrak{M}_{sr\odot\oplus}$. Moreover, by monotonicity, $\Gamma \vDash_{\mathfrak{M}} \Delta$.

($\Leftarrow$) Suppose that $\Gamma \vDash_{\mathfrak{M}} \Delta$ and that either $\varnothing \vDash_{\mathfrak{M}} \Delta$ or there is some non-empty $\Gamma' \subseteq \Gamma$ such that $\Gamma \vDash_{\mathfrak{M}} \Delta$ and both $Var^+(\Gamma') \subseteq Var^+(\Delta)$ and $Var^-(\Gamma') \subseteq Var^-(\Delta)$.

- First assume that $\varnothing \vDash_{\mathfrak{M}} \Delta$—which implies that $\Gamma \vDash_{\mathfrak{M}} \Delta$. Then, for all valuations $v \in \text{Hom}_{\mathfrak{M}}$, $v(\delta) \in D$ for some $\delta \in \Delta$. This, together with the fact that the left coordinate of any valuation over $\mathsf{A} \times \mathsf{IS}_4$ behaves as a valuation over A, also ensures that there is no $v' \in \text{Hom}_{\mathfrak{M}_{sr\odot\oplus}}$ such that $v'(\delta) \in \overline{D} \times \{\odot, \oplus, \ominus, \otimes\}$ for all $\delta \in \Delta$, or else there would be a valuation over $\mathfrak{M}$ assigning $\overline{D}$ to all $\delta \in \Delta$, contradicting the assumption. Therefore, $\varnothing \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$ and by monotonicity $\Gamma \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$.

- Second, assume that that there is some non-empty $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vDash_{\mathfrak{M}} \Delta$ and both $Var^+(\Gamma') \subseteq Var^+(\Delta)$ and $Var^-(\Gamma') \subseteq Var^-(\Delta)$—which guarantees by monotonicity that $\Gamma \vDash_{\mathfrak{M}} \Delta$. To establish that $\Gamma' \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$, so again by monotonicity that $\Gamma \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$, suppose that some $v' \in \text{Hom}_{\mathfrak{M}_{sr\odot\oplus}}$ is such that $v'(\gamma) \notin \overline{D} \times \{\odot, \oplus\}$ for all $\gamma \in \Gamma'$.

  There are two cases: if $v'(\gamma) \in D \times \{\odot, \oplus, \ominus, \otimes\}$ for all $\gamma \in \Gamma'$, then the assumption that $\Gamma' \vDash_{\mathfrak{M}} \Delta$ guarantees that $v'(\delta) \in D \times \{\odot, \oplus, \ominus, \otimes\}$ for some $\delta \in \Delta$. If, instead, there is some $\gamma \in \Gamma'$ such that $v'(\gamma) \in \overline{D} \times \{\ominus, \oplus\}$, then the assumption that $Var^+(\Gamma') \subseteq Var^+(\Delta)$ and $Var^-(\Gamma') \subseteq Var^-(\Delta)$, thus that for all $\gamma \in \Gamma'$ also $Var^+(\gamma) \subseteq Var^+(\Delta)$ and $Var^-(\gamma) \subseteq Var^-(\Delta)$, guarantees, by Fact 4.2.2, that $v'(\delta) \in A \times \{\ominus, \otimes\}$. In any case, since v' was arbitrary, $\Gamma' \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$, thus $\Gamma \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$.

□

An analogous strategy is employed in the proof of the following characterization.

Theorem 4.3.2.

$$\Gamma \vDash_{\mathfrak{M}_{s\odot\oplus}} \Delta \textit{ iff } \Gamma \vDash_{\mathfrak{M}} \Delta \textit{ and } \begin{cases} \textit{(i) } \Gamma \vDash_{\mathfrak{M}} \varnothing \textit{ or} \\ \textit{(ii) } \exists \Delta' \mid \Delta' \subseteq \Delta \textit{ and } \Delta' \neq \varnothing \textit{ such that } \Gamma \vDash_{\mathfrak{M}} \Delta' \\ \textit{ and } Var^{+}(\Delta') \subseteq Var^{+}(\Gamma),\ Var^{-}(\Delta') \subseteq Var^{-}(\Gamma) \end{cases}$$

Hence, moving on, the already mentioned correspondence can be proved[7].

Theorem 4.3.3. *For all* $\Gamma, \Delta \subseteq Fml(\mathscr{L})$*:*

(i) $\Gamma \vDash_{\mathfrak{M}_{s\odot}} \Delta$ *if and only if* $\Gamma \vDash_{\mathfrak{M}^{r}[e]} \Delta$

(ii) $\Gamma \vDash_{\mathfrak{M}_{sr\odot}} \Delta$ *if and only if* $\Gamma \vDash_{\mathfrak{M}^{l}[e]} \Delta$

(iii) $\Gamma \vDash_{\mathfrak{M}_{s\odot\oplus}} \Delta$ *if and only if* $\Gamma \vDash_{\mathfrak{M}^{r}[\pm]} \Delta$

(iv) $\Gamma \vDash_{\mathfrak{M}_{sr\odot\oplus}} \Delta$ *if and only if* $\Gamma \vDash_{\mathfrak{M}^{l}[\pm]} \Delta$

Proof. Each point can be proved showing that counterexamples to an inference over one matrix can be turned to counterexamples in the other and vice versa.

For instance, consider point (i). Suppose that $\Gamma \nvDash_{\mathfrak{M}^{r}[e]} \Delta$. Hence there is $v \in \text{Hom}_{\mathfrak{M}^{r}[e]}$ such that $v(\gamma) \in D$ for all $\gamma \in \Gamma$ while $v(\delta) \in (\overline{D} \cup \{e\})$. Now construct a valuation $v' \in \text{Hom}_{\mathfrak{M}_{s\odot}}$ such that if $v(q) = a$ for $a \neq e$, then $v'(q) = \langle a, \odot \rangle$, otherwise $v'(q) = \langle b, \otimes \rangle$ for any fixed $b \in A$. Since v' extends to complex formulas on the left coordinate just as v when restricted to values in A, for any ψ, if $v(\psi) = a$ for some $a \in A$, then $v'(\psi) = \langle a, \odot \rangle$, while if $v(\psi) = e$ then by Fact 4.2.1, $v'(\psi) = \langle c, \otimes \rangle$ for some $c \in A$. Hence $v'(\gamma) \in (D \times \{\odot\})$ for all $\gamma \in \Gamma$ while $v(\delta) \notin (D \times \{\odot\})$ for all $\delta \in \Delta$. Therefore, $\Gamma \nvDash_{\mathfrak{M}_{s\odot}} \Delta$.

[7] Points (i) and (ii) below, when restricted to single-conclusion inferences, are straightforward corollaries to Theorem 5.1.7, Remark 5.1.10, and Theorem 6.1.8 in Bonzio et al., 2022.

For the other direction, suppose that $\Gamma \nvDash_{\mathfrak{M}_{s\odot}} \Delta$, so there is $v' \in \mathrm{Hom}_{\mathfrak{M}_{s\odot}}$ such that $v'(\gamma) \in (D \times \{\odot\})$ for all $\gamma \in \Gamma$ while $v(\delta) \notin (D \times \{\odot\})$ for all $\delta \in \Delta$. Then construct $v \in \mathrm{Hom}_{\mathfrak{M}^r[e]}$ such that if $v'(q) = \langle a, \odot \rangle$ then $v(q) = a$ while if $v'(q) = \langle a, \otimes \rangle$ then $v(q) = e$. For the same reason as before, for complex formulas ψ, if $v'(\psi) = \langle a, \odot \rangle$ then $v(\psi) = a$ while if $v'(\psi) = \langle a, \otimes \rangle$ then by Fact 2.3.1, $v(\psi) = e$. Therefore, $v(\gamma) \in D$ for all $\gamma \in \Gamma$ while $v(\delta) \in (\overline{D} \cup \{e\})$, implying that $\Gamma \nvDash_{\mathfrak{M}^r[e]} \Delta$. The same constructions can be used for point (ii).

For points (iii)-(iv) the strategy is similar. When there is a counterexample v over the balanced (left or right) infectious extension, construct a valuation v' over $\mathsf{A} \times \mathsf{IS}_4$ such that if $v(p) = \langle e, e \rangle$, $v'(p) = \langle a, \otimes \rangle$ for some fixed $a \in A$, if, for some $b \in A$, $v(p) = \langle b, e \rangle$ then $v'(p) = \langle b, \oplus \rangle$, while if $v(p) = \langle e, b \rangle$ then $v'(p) = \langle \neg b, \ominus \rangle$ and, finally, if $v(p) = b$ then $v'(p) = \langle b, \odot \rangle$. The other direction of the proofs builds on the opposite construction. □

This family of results has at least two immediate consequences. First, particularly Theorems 4.2.5, 4.2.6 and 4.3.3, outline a correspondence between the infectious value(s) both in a regular and in a balanced framework, suggesting a conceptual bridge for relating meaninglessness—discussed in section 2.5—and on/off-topicality that was partly mentioned at the beginning of this chapter. Consider the regular case: the infectious value is mapped to any two-address value with the off-topic right-coordinate and vice versa, all such off-topic values are mapped to the contaminant value. This one-to-many or many-to-one mapping, contrary to the one-to-one correspondence between on-topic values and the respective alethic value, shows that under the meaningless reading there is only one way to be nonsense, while the ways to be meaningful are as many as the alethic statuses. Instead, the on/off-topic interpretation makes explicit all ways in which a statement may be meaningless, that is, all the off-topic values, which this time are equinumerous with the ways of being meaningful, that is, on-topic.

The balanced case is similar but distinguishes between statements which are *fully* meaningless, *fully* meaningful and statements that are *partly* both, *i.e.* meaningless but their negation meaningful or vice versa. In this case once more only statements which are at least partly meaningful can be so in different ways, being assigned different alethic values. But whenever both a statement and its negation are meaningless, the appropriate *fully* meaningless value will be assigned with no further options. The direct product construction instead concedes the same number of ways of being in one way or the other, providing the same treatment for fully or partly meaningful or meaningless statements. One advantage of this richer framework will be discussed below with respect to the flexibility of the two-address strategy to accommodate different kinds of logical consequences.

For the second consequence of the above result, consider Theorems 2.4.1 and 2.4.2 as well as Theorems 2.4.3 and 2.4.4. With the above correspondences and these characterization results, the simultaneous strategy constitutes another way to build matrix semantics for weak (balanced) variable inclusion companions of a given structural logic—extending the result to classes of matrices in the expected way. Therefore, all the considerations applying to those companions also apply to this case. Namely, whenever, a logic has theorems, so will its weak (balanced) left companion and if it has antitheorems, so will its weak (balanced) right companion. This, with the monotonicity of the consequence relations, opens the door for the exceptions to relevance that were contested in chapter 3. But just as, in a structural framework, pure variable inclusion logics by Paoli et al., 2021 offer an alternative—possibly a solution—by employing a matrix which takes care of preservation of meaningfulness or meaninglessness, the parallel strategy outlined in the previous section can be implemented in the same way for this case.

Given all the characterizations presented so far, it is now possible to discuss one of the advantages of the two-address framework anticipated previously with respect to its flexibility. To illustrate it and also to clarify the differences between the simultaneous and the parallel strategies, consider the settings discussed in section 4.1 which can be modeled on the alethic side by B_2, K_3 or DM_4 and on the topical side by IS_2 or IS_4. In the following, $\mathbf{L}_{\mathfrak{M}}$ indicates the logic induced by $\mathfrak{M}^{\mathbf{L}}$. Moreover, given a matrix $\mathfrak{M}^{\mathbf{L}} = \langle \mathsf{A}, D \rangle$, $\mathfrak{M}^{\mathbf{L}}_{a} = \langle \mathsf{A} \times \mathsf{IS}_2, D \times \{\odot, \otimes\} \rangle$ while $\mathfrak{M}^{\mathbf{L}}_{ab} = \langle \mathsf{A} \times \mathsf{IS}_4, D \times \{\odot, \otimes, \oplus, \ominus\} \rangle$—namely, the last two coincide with the first on the choice of designated values as the topical coordinate is immaterial, thus the a subscript standing for the focus on the alethic coordinate and ab when the topical structure is IS_4. Some of the following results were proved in *e.g.* Bonzio et al., 2022[8], Szmuc, 2017, Randriamahazaka, 2022, Fine, 2016a.

$$\mathbf{CL} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{\mathbf{a}}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{\mathbf{ab}}} \qquad \mathbf{LP} = \mathbf{L}_{\mathfrak{M}^{\mathbf{LP}}_{a}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{LP}}_{ab}}$$

$$\mathbf{K_3} = \mathbf{L}_{\mathfrak{M}^{\mathbf{K_3}}_{a}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{K_3}}_{ab}} \qquad \mathbf{BD} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{a}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{ab}}$$

$$\mathbf{CL^{wr}} = \mathbf{K_3^{w}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{\mathbf{s}\odot}} \qquad \mathbf{CL^{wl}} = \mathbf{PWK} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{\mathbf{sr}\odot}}$$

$$\mathbf{CL^{pr}} = \mathbf{PAI}_{\mathtt{fde}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{\boldsymbol{p}\odot}} \qquad \mathbf{CL^{pl}} = \mathbf{DD}_{\mathtt{fde}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{\boldsymbol{pr}\odot}}$$

[8]See in particular the Table in Bonzio et al., 2022, p.133

$$\mathbf{LP^{wr}} = \mathbf{LP^{pr}} = \mathbf{S}_{\mathtt{fde}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{LP}}_{s\odot}} \qquad \mathbf{LP^{wl}} = \mathbf{S}_{\mathtt{nfl}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{LP}}_{sr\odot}}$$

$$\mathbf{K_3}^{\mathbf{wr}} = \mathbf{S}_{\mathtt{etl}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{K_3}}_{s\odot}} \qquad \mathbf{K_3}^{\mathbf{wl}} = \mathbf{K_3}^{\mathbf{pl}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{K_3}}_{sr\odot}}$$

$$\mathbf{LP^{pl}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{LP}}_{pr\odot}} \qquad \mathbf{K_3}^{\mathbf{pr}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{SK}}_{p\odot}}$$

$$\mathbf{BD^{wr}} = \mathbf{BD^{pr}} = \mathbf{S}^{*}_{\mathtt{fde}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{s\odot}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{p\odot}}$$

$$\mathbf{BD^{wl}} = \mathbf{BD^{pl}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{pr\odot}}$$

$$\mathbf{CL^{wr\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{s\odot\oplus}} \qquad \mathbf{CL^{pr\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{p\odot\oplus}}$$

$$\mathbf{CL^{wl\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{sr\odot\oplus}} \qquad \mathbf{CL^{pl\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{CL}}_{pr\odot\oplus}}$$

$$\mathbf{LP^{wr\pm}} = \mathbf{LP^{pr\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{LP}}_{s\odot\oplus}} \qquad \mathbf{LP^{wl\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{LP}}_{sr\odot\oplus}}$$

$$\mathbf{K_3}^{\mathbf{wr\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{K_3}}_{s\odot\oplus}} \qquad \mathbf{K_3}^{\mathbf{wl\pm}} = \mathbf{K_3}^{\mathbf{pl\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{K_3}}_{sr\odot\oplus}}$$

$$\mathbf{LP^{pl\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{LP}}_{pr\odot\oplus}} \qquad \mathbf{K_3}^{\mathbf{pr\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{SK}}_{p\odot\oplus}}$$

$$\mathbf{BD^{wr\pm}} = \mathbf{BD^{pr\pm}} = \mathbf{AC}_{\mathtt{fde}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{s\odot\oplus}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{p\odot\oplus}}$$

$$\mathbf{BD^{wl\pm}} = \mathbf{BD^{pl\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{pr\odot\oplus}}$$

This list shows that keeping the same algebraic structure and only changing the designated set of values, **CL**, **LP**, $\mathbf{K_3}$ and **BD** as well as their respective companions can be obtained, which was not possible in other frameworks, such as the three-valued one[9]. In particular, observe that whenever truth *simpliciter*, regardless of the topic status, is preserved, one obtains **CL**, if no truth gaps or gluts are admitted, $\mathbf{K_3}$ or **LP** if one of them is, or **BD**, if truth and falsity are neither exclusive nor exhaustive.

[9]Unless mixed standards for premises and conclusions are employed, in particular using *p*-matrices as in Szmuc and Ferguson, 2021.

This suggests, as done in Song et al., 2023, a reading of their consequence relations as *topic neutral*, since their notion of validity does not distinguish between topics, or, if it does, such distinctions play no role. Instead, all definitions of validity that draw such distinction, for instance demanding simultaneous or parallel preservation of truth and topic, can be understood as *topic-sensitive* consequence relations, as discussed in chapter 1.

The next section will further investigate the flexibility of this framework, presenting a novel proposal which also points to further ways in which its potential may be leveraged.

4.4 Non-deterministic Two-address Semantics

In addition to the considerations above, the two-address setting exhibits versatility in a further respect, that is, as a non-deterministic semantic framework capable of capturing all the logics discussed above with fewer semantic values. This is shown in Szmuc and Zirattu, 2025 where the results are limited to single-conclusion inferences and right companions. The present section extends their work to multiple-conclusion inferences and left companions as well.

Recall that in section 4.1, the use of certain algebras was justified as a way of reifying specific alethic or topical statuses. However, introducing additional values for gappy or glutty statements, or for statements whose topic shifts under negation, is not the only way to model such statuses. An alternative approach is to liberalize the behavior of value assignments. This path can be pursued for each coordinate of the two-address valuations, separately or at the same time, and could potentially involve all logical operators, though, for the purposes of this work, it will only be restricted to negation. Even in this case though, some restrictions will be needed to ensure control over the assignments for negated formulas. In particular, given that in all of the systems considered negation is involutive and the De Morgan laws hold, the following assumptions will be kept for any valuation v and any $\varphi, \psi \in Fml(\mathscr{L})$ appearing from now on—as in Batens et al., 1999, Rosenblatt, 2015 and Szmuc and Zirattu, 2025.

(**A1**) $v(\psi) = v(\neg\neg\psi)$

(**A2**) $v(\neg(\psi \wedge \varphi)) = v(\neg\psi \vee \neg\varphi)$

(**A3**) $v(\neg(\psi \vee \varphi)) = v(\neg\psi \wedge \neg\varphi)$

Truth and falsity being non-exhaustive and non-exclusive may be represented as the *possibility* of a statement and its negation being both false or both true, respectively. Thus, it *may* happen that the negation of a true statement is false but *may* also remain true, and likewise the negation of a false statement *may* be true or *may* remain false. This relaxation of the classical assumptions governing truth and falsity can be captured relaxing the clauses describing the evaluation of the alethic status of negated formulas (see point 1a in section 4.1). This allows to "mimic" the glutty and gappy values which, this time, need not be added explicitly as part of the universe of evaluation, but rather correspond to specific ways of assigning the usual Boolean values when non-determinism is allowed.

This idea is also applicable to the topical values. The topic transformativeness of negation could be depicted adding specific values for those cases in which a statement and its negation differ in topic. However, recall that the account of topic transformation—see section 1.6—captured by IS_4—see section 4.1—is itself grounded in the idea that negation *may* change the topic of a statement, but need not do so. Therefore, this case appears to be even more naturally handled within a non-deterministic framework in which the clauses governing the evaluation of the topical status of negated statements are relaxed (see point 1b in section 4.1). Hence, as in the alethic case, the negation of an on-topic statement may be itself on topic or not, while the negation of an off-topic statement may be off-topic too or not.

Once more, a key advantage of the two-address framework is that it allows independent choices for alethic and topical evaluations. Similarly to the deterministic case where algebras for each coordinate can be selected freely, non-determinism can be permitted with different degrees only for one aspect or the other, or both. This fine-grained control, unavailable in other semantic frameworks—where, for example, there is only one way to be meaningless (see section 4.2)—demonstrates the approach's flexibility.

Moving to the formalities, some new notions need to be introduced. First, a notion which generalizes that of (deterministic) logical matrix in Definition 2.2.4.

Definition 4.4.1. A *logical Nmatrix* $\mathfrak{M}$ for $\mathscr{L}$ is a tuple $\mathfrak{M} = \langle A, D, \mathrm{O} \rangle$, where A is a non-empty set of truth values, D is a non-empty subset of A, and for every n-ary connective $\star$ of $\mathscr{L}$, O includes a corresponding function $\tilde{\star} : A^n \to 2^A \setminus \{\varnothing\}$.

In this definition, the tuple $\langle A, \mathrm{O} \rangle$ is a *multialgebra*, also called *hyperalgebra* or simply non-deterministic algebra (see M. E. Coniglio et al., 2020 for more details). Therefore, Nmatrices can be seen as multialgebras of type $\mathscr{L}$ equipped with a set of designated values. It is also possible to define valuations over Nmatrices in a way that generalizes the notion of deterministic valuations in Definition 2.2.5.

Definition 4.4.2. A *dynamic valuation based on a Nmatrix* $\mathfrak{M} = \langle A, D, \mathrm{O}\rangle$ is a function v from the set of formulas $Fml(\mathscr{L})$ to A, which is closed under subformulas, such that for each n-ary connective $\star$ of $\mathscr{L}$, and for all $\pi_1, \ldots, \pi_n \in Fml(\mathscr{L})$ it holds that $v(\star(\pi_1, \ldots, \pi_n)) \in \tilde{\star}(v(\pi_1), \ldots, v(\pi_n))$.

Finally, just as regular logical matrices, also Nmatrices induce structural consequence relations in the expected manner: $\Gamma \vDash_{\mathfrak{M}} \Delta$ if and only if for all dynamic valuations v over $\mathfrak{M}$, if $v(\gamma) \in D$ for all $\gamma \in \Gamma$, then $v(\delta) \in D$ for some $\delta \in \Delta$. Similarly, a class of Nmatrices $\boldsymbol{\mathfrak{M}}$ induces a consequence relation $\vDash_{\boldsymbol{\mathfrak{M}}}$ by letting $\Gamma \vDash_{\boldsymbol{\mathfrak{M}}} \Delta$ if and only if $\Gamma \vDash_{\mathfrak{M}} \Delta$ for all $\mathfrak{M} \in \boldsymbol{\mathfrak{M}}$.

Having set the technical notions, it is now possible to present—following Szmuc and Zirattu, 2025—the formal counterpart to the above idea of a non-deterministic negation which may be so only for the alethic coordinate, only for the topical coordinate, or for both. Consider first the negations displayed in Figure 4.2. They are non-deterministic only on the alethic coordinate—hence the a superscript—the first pair only mimicking the glutty status since the negation of a true statement may itself be true but that of a false statement can only be true—hence the gl subscript—the second only representing the gappy status for the dual reason—hence the gp subscript—the third modeling both as it relaxes all restrictions. Notice that in any case the same degree of alethic non-determinism can be combined with a (deterministic) topic transparent behavior of negation as well as with a topic-transformative one—indicated by the $\pm$ superscript—which is modeled by IS_2 or IS_4 respectively.

Second, consider the negations in Figure 4.3. They are all non-deterministic

	$\neg^a_{\mathrm{gl}}$
$t\odot$	$\{f\odot, t\odot\}$
$t\otimes$	$\{f\otimes, t\otimes\}$
$f\odot$	$\{t\odot\}$
$f\otimes$	$\{t\otimes\}$

	$\neg^{a\pm}_{\mathrm{gl}}$
$t\odot$	$\{f\odot, t\odot\}$
$t\oplus$	$\{f\ominus, t\ominus\}$
$t\ominus$	$\{f\oplus, t\oplus\}$
$t\otimes$	$\{f\otimes, t\otimes\}$
$f\odot$	$\{t\odot\}$
$f\oplus$	$\{t\ominus\}$
$f\ominus$	$\{t\oplus\}$
$f\otimes$	$\{t\otimes\}$

	$\neg^a_{\mathrm{gp}}$
$t\odot$	$\{f\odot\}$
$t\otimes$	$\{f\otimes\}$
$f\odot$	$\{t\odot, f\odot\}$
$f\otimes$	$\{t\otimes, f\otimes\}$

	$\neg^{a\pm}_{\mathrm{gp}}$
$t\odot$	$\{f\odot\}$
$t\oplus$	$\{f\ominus\}$
$t\ominus$	$\{f\oplus\}$
$t\otimes$	$\{f\otimes\}$
$f\odot$	$\{t\odot, f\odot\}$
$f\oplus$	$\{t\ominus, f\ominus\}$
$f\ominus$	$\{t\oplus, f\oplus\}$
$f\otimes$	$\{t\otimes, f\otimes\}$

	$\neg^a_{\mathrm{glgp}}$
$t\odot$	$\{f\odot, t\odot\}$
$t\otimes$	$\{f\otimes, t\otimes\}$
$f\odot$	$\{t\odot, f\odot\}$
$f\otimes$	$\{t\otimes, f\otimes\}$

	$\neg^{a\pm}_{\mathrm{glgp}}$
$t\odot$	$\{f\odot, t\odot\}$
$t\oplus$	$\{f\ominus, t\ominus\}$
$t\ominus$	$\{f\oplus, t\oplus\}$
$t\otimes$	$\{f\otimes, t\otimes\}$
$f\odot$	$\{t\odot, f\odot\}$
$f\oplus$	$\{t\ominus, f\ominus\}$
$f\ominus$	$\{t\oplus, f\oplus\}$
$f\otimes$	$\{t\otimes, f\otimes\}$

Figure 4.2 Alethic non-deterministic negations

only on the topical side—thus the t superscript—and each of them displays a different set of (deterministic) values for the alethic coordinate. The first only counts classical truth and falsity, the second has one extra value—which, without loss of generality, is taken to be glutty, hence the *gl* subscript—while the third includes two extra values—thus the *glgp* subscript. Notice that in this case, unlike the previous, non-determinism for the topical side does not come with degrees or else—as shown below—it would not be possible to model the target topic-transformative behavior of negation which, so far, has only been represented by IS_4. Moreover, observe that if non-determinism were only allowed in such a way that, *e.g.*, the negation of an on-topic statement is always on-topic while that of an off-topic statement may be on or off-topic—analogously to what happens when only gappy or glutty alethic values are represented non-deterministically—

then imposing (**A1**) would filter down the set of such dynamic valuations over IS_2 to the deterministic ones that assign the off-topic value only to the negation of an off-topic formula. If that were not the case, then it could happen that φ receives value $\otimes$, but $\neg\neg\varphi$ is assigned $\odot$ because $\neg\varphi$ has value $\odot$. This shows that the implementation over IS_2 of non-determinism which respects the assumptions above cannot proceed in this graded manner. There could be, though, another way to achieve non-determinism of negation only for one between $\odot$ and $\otimes$ that is compatible with (**A1**)-(**A3**). That is, keeping the negation of $\odot$ as $\otimes$ only, while the negation of $\otimes$ may be $\otimes$ or $\odot$—or vice-versa. Although this alternative is legitimate, it cannot work to capture the logics considered here as the behavior of negation would mimic the values of another topical structure other than IS_4. This point, though, remains open for another discussion.

	$\neg^t$
$t\odot$	$\{f\odot, f\otimes\}$
$t\otimes$	$\{f\odot, f\otimes\}$
$f\odot$	$\{t\odot, t\otimes\}$
$f\otimes$	$\{t\odot, t\otimes\}$

	$\neg^t_{\text{gl}}$
$t\odot$	$\{f\odot, f\otimes\}$
$t\otimes$	$\{f\odot, f\otimes\}$
$i\odot$	$\{i\odot, i\otimes\}$
$i\otimes$	$\{i\odot, i\otimes\}$
$f\odot$	$\{t\odot, t\otimes\}$
$f\otimes$	$\{t\odot, t\otimes\}$

	$\neg^t_{\text{glgp}}$
$t\odot$	$\{f\odot, f\otimes\}$
$t\otimes$	$\{f\odot, f\otimes\}$
$b\odot$	$\{b\odot, b\otimes\}$
$b\otimes$	$\{b\odot, b\otimes\}$
$n\odot$	$\{n\odot, n\otimes\}$
$n\otimes$	$\{n\odot, n\otimes\}$
$f\odot$	$\{t\odot, t\otimes\}$
$f\otimes$	$\{t\odot, t\otimes\}$

Figure 4.3 Topical non-deterministic negations

Finally, consider the negations displayed in Figure 4.4. They are non-deterministic on both sides—hence the *at* superscript—all in the same way for the topical coordinate, the first only mimicking the glutty value, the second only the gappy and the third both—hence the subscripts.

The introduction of the non-deterministic negations above allows then to define the Nmatrices which will be employed in the upcoming semantic

	$\neg^{at}_{\text{gl}}$
$t\odot$	$\{f\odot, f\otimes, t\odot, t\otimes\}$
$t\otimes$	$\{f\odot, f\otimes, t\odot, t\otimes\}$
$f\odot$	$\{t\odot, t\otimes\}$
$f\otimes$	$\{t\odot, t\otimes\}$

	$\neg^{at}_{\text{gp}}$
$t\odot$	$\{f\odot, f\otimes\}$
$t\otimes$	$\{f\odot, f\otimes\}$
$f\odot$	$\{t\odot, t\otimes, f\odot, f\otimes\}$
$f\otimes$	$\{t\odot, t\otimes, f\odot, f\otimes\}$

	$\neg^{at}_{\text{glgp}}$
$t\odot$	$\{f\odot, f\otimes, t\odot, t\otimes\}$
$t\otimes$	$\{f\odot, f\otimes, t\odot, t\otimes\}$
$f\odot$	$\{t\odot, t\otimes, f\odot, f\otimes\}$
$f\otimes$	$\{t\odot, t\otimes, f\odot, f\otimes\}$

Figure 4.4 Alethic and topical non-deterministic negations

characterizations. These characterizations will be explicitly provided only for logics of variable inclusion, leaving the consequence relations previously dubbed as *topic neutral*—*i.e.* **CL**, $\mathbf{K_3}$, **LP**, **BD**—aside, as they can be captured employing the same strategy described in the previous section, namely designating the appropriate set of designated values irrespective of the topical coordinate. The first non-deterministic structures to be introduced replace the negation function of $\mathsf{B}_2 \times \mathsf{IS}_2$ or $\mathsf{B}_2 \times \mathsf{IS}_4$ with one in Figure 4.2.

Definition 4.4.3.

- Let $\mathfrak{M}^a_{s\odot,gl}$, $\mathfrak{M}^a_{s\odot,gp}$ and $\mathfrak{M}^a_{s\odot,glgp}$ be the Nmatrices obtained replacing the negation function of $\mathfrak{M}^{\mathbf{CL}}_{s\odot}$ with $\neg^a_{\text{gl}}$, $\neg^a_{\text{gp}}$ and $\neg^a_{\text{glgp}}$, respectively.

- Let $\mathfrak{M}^a_{sr\odot,gl}$, $\mathfrak{M}^a_{sr\odot,gp}$ and $\mathfrak{M}^a_{sr\odot,glgp}$ be the Nmatrices obtained replacing the negation function of $\mathfrak{M}^{\mathbf{CL}}_{sr\odot}$ with $\neg^a_{\text{gl}}$, $\neg^a_{\text{gp}}$ and $\neg^a_{\text{glgp}}$, respectively.

- Let $\boldsymbol{\mathfrak{M}}^a_{p\odot,gl}$, $\boldsymbol{\mathfrak{M}}^a_{p\odot,gp}$ and $\boldsymbol{\mathfrak{M}}^a_{p\odot,glgp}$ be the classes of Nmatrices obtained replacing the negation function of $\boldsymbol{\mathfrak{M}}^{\mathbf{CL}}_{p\odot}$ with $\neg^a_{\text{gl}}$, $\neg^a_{\text{gp}}$ and $\neg^a_{\text{glgp}}$, respectively.

- Let $\boldsymbol{\mathfrak{M}}^a_{pr\odot,gl}$, $\boldsymbol{\mathfrak{M}}^a_{pr\odot,gp}$ and $\boldsymbol{\mathfrak{M}}^a_{pr\odot,glgp}$ be the classes of Nmatrices obtained replacing the negation function of $\boldsymbol{\mathfrak{M}}^{\mathbf{CL}}_{pr\odot}$ with $\neg^a_{\text{gl}}$, $\neg^a_{\text{gp}}$ and $\neg^a_{\text{glgp}}$, respectively.

- Let $\mathfrak{M}^{a\pm}_{s\odot\oplus,gl}$, $\mathfrak{M}^{a\pm}_{s\odot\oplus,gp}$ and $\mathfrak{M}^{a\pm}_{s\odot\oplus,glgp}$ be the Nmatrices obtained replacing the negation function of $\mathfrak{M}^{\mathbf{CL}}_{s\odot\oplus}$ with $\neg^{a\pm}_{\text{gl}}$, $\neg^{a\pm}_{\text{gp}}$ and $\neg^{a\pm}_{\text{glgp}}$, respectively.

- Let $\mathfrak{M}^{a\pm}_{sr\odot\oplus,gl}$, $\mathfrak{M}^{a\pm}_{sr\odot\oplus,gp}$ and $\mathfrak{M}^{a\pm}_{sr\odot\oplus,glgp}$ be the Nmatrices obtained replacing the negation function of $\mathfrak{M}^{\mathbf{CL}}_{sr\odot\oplus}$ with $\neg^{a\pm}_{\text{gl}}$, $\neg^{a\pm}_{\text{gp}}$ and $\neg^{a\pm}_{\text{glgp}}$, respectively.

- Let $\boldsymbol{\mathfrak{M}}^{a\pm}_{p\odot\oplus,gl}$, $\boldsymbol{\mathfrak{M}}^{a\pm}_{p\odot\oplus,gp}$ and $\boldsymbol{\mathfrak{M}}^{a\pm}_{p\odot\oplus,glgp}$ be the classes of Nmatrices obtained replacing the negation function of $\boldsymbol{\mathfrak{M}}^{\mathbf{CL}}_{p\odot\oplus}$ with $\neg^{a\pm}_{\text{gl}}$, $\neg^{a\pm}_{\text{gp}}$ and $\neg^{a\pm}_{\text{glgp}}$, respectively.

- Let $\boldsymbol{\mathfrak{M}}^{a\pm}_{pr\odot\oplus,gl}$, $\boldsymbol{\mathfrak{M}}^{a\pm}_{pr\odot\oplus,gp}$ and $\boldsymbol{\mathfrak{M}}^{a\pm}_{pr\odot\oplus,glgp}$ be the classes of Nmatrices obtained replacing the negation function of $\boldsymbol{\mathfrak{M}}^{\mathbf{CL}}_{pr\odot\oplus}$ with $\neg^{a\pm}_{\text{gl}}$, $\neg^{a\pm}_{\text{gp}}$ and $\neg^{a\pm}_{\text{glgp}}$, respectively.

As in the previous section, $\mathbf{L}_{\mathfrak{M}}$ ($\mathbf{L}_{\boldsymbol{\mathfrak{M}}}$) denotes the logic induced by a matrix $\mathfrak{M}$ (a class of matrices $\boldsymbol{\mathfrak{M}}$). Hence, the following equalities can be proved.

$$\mathbf{S}_{\texttt{fde}} = \mathbf{L}_{\mathfrak{M}^{a}_{s\odot,gl}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{a}_{p\odot,gl}} \qquad \mathbf{S}_{\texttt{etl}} = \mathbf{L}_{\mathfrak{M}^{a}_{s\odot,gp}}$$

$$\mathbf{K_3}^{\mathbf{pr}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{a}_{p\odot,gp}} \qquad \mathbf{S}^{*}_{\texttt{fde}} = \mathbf{L}_{\mathfrak{M}^{a}_{s\odot,glgp}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{a}_{p\odot,glgp}}$$

$$\mathbf{S}_{\texttt{nfl}} = \mathbf{L}_{\mathfrak{M}^{a}_{sr\odot,gl}} \qquad \mathbf{K_3}^{\mathbf{wl}} = \mathbf{L}_{\mathfrak{M}^{a}_{sr\odot,gp}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{a}_{pr\odot,gp}}$$

$$\mathbf{LP}^{\mathbf{pl}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{a}_{pr\odot,gl}} \qquad \mathbf{BD}^{\mathbf{wl}} = \mathbf{L}_{\mathfrak{M}^{a}_{sr\odot,glgp}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{a}_{pr\odot,glgp}}$$

$$\mathbf{LP}^{wr\pm} = \mathbf{L}_{\mathfrak{M}^{a\pm}_{s\odot\oplus,gl}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{a\pm}_{p\odot\oplus,gl}} \qquad \mathbf{K_3}^{\mathbf{wr}\pm} = \mathbf{L}_{\mathfrak{M}^{a\pm}_{s\odot\oplus,gp}}$$

$$\mathbf{K_3}^{\mathbf{pr}\pm} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{a\pm}_{p\odot\oplus,gp}} \qquad \mathbf{AC}_{\texttt{fde}} = \mathbf{L}_{\mathfrak{M}^{a\pm}_{s\odot\oplus,glgp}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{a\pm}_{p\odot\oplus,glgp}}$$

$$\mathbf{LP}^{\mathbf{wl}\pm} = \mathbf{L}_{\mathfrak{M}^{a\pm}_{sr\odot\oplus,gl}} \qquad \mathbf{K_3}^{\mathbf{wl}\pm} = \mathbf{L}_{\mathfrak{M}^{a\pm}_{sr\odot\oplus,gp}} = \mathbf{L}_{\mathfrak{M}^{a}_{pr\odot\oplus,gp}}$$

$$\mathbf{LP}^{\mathbf{pl}\pm} = \mathbf{L}_{\mathfrak{M}^{a\pm}_{pr\odot\oplus,gl}} \qquad \mathbf{BD}^{\mathbf{wl}\pm} = \mathbf{L}_{\mathfrak{M}^{a\pm}_{sr\odot\oplus,glgp}} = \mathbf{L}_{\mathfrak{M}^{a\pm}_{pr\odot\oplus,glgp}}$$

To streamline the exposition, only the equality pertaining to $\mathbf{BD}^{\mathbf{wl}\pm}$ is proved. The remaining results for the left variable inclusion companions follow as straightforward restrictions, while the proof for $\mathbf{AC}_{\mathtt{fde}}$ relies on the same construction—also used in Szmuc and Zirattu, 2025, Fact 4.1—with the other right companions obtained as its restrictions.

Theorem 4.4.1. $\Gamma \vDash_{\mathfrak{M}^{a\pm}_{sr\odot\oplus,glgp}} \Delta$ *if and only if* $\Gamma \vdash_{\mathbf{BD}^{\mathbf{wl}\pm}} \Delta$

Proof. In section 4.3 it was proved that $\mathbf{BD}^{\mathbf{wl}\pm} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{sr\oplus\odot}}$. Therefore it is sufficient to prove that $\Gamma \vDash_{\mathfrak{M}^{a\pm}_{sr\odot\oplus,glgp}} \Delta$ if and only if $\Gamma \vDash_{\mathfrak{M}^{\mathbf{BD}}_{sr\oplus\odot}} \Delta$. Both directions are proved by contraposition.

(⇒) Suppose that $\Gamma \nvDash_{\mathfrak{M}^{\mathbf{BD}}_{sr\oplus\odot}} \Delta$. So there is $v \in \mathrm{Hom}_{\mathfrak{M}^{\mathbf{BD}}_{sr\oplus\odot}}$ such that $v(\gamma) \notin \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\gamma \in \Gamma$, but $v(\delta) \in \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\delta \in \Delta$. Then construct a valuation v' over $\mathsf{B}_2 \times \mathsf{IS}_4$ such that for all literals $p, \neg p$:

$$v'(p) = \begin{cases} t\odot & \text{if } v(p) = t\odot \text{ or } v(p) = b\odot \\ t\oplus & \text{if } v(p) = t\oplus \text{ or } v(p) = b\oplus \\ t\ominus & \text{if } v(p) = t\ominus \text{ or } v(p) = b\ominus \\ t\otimes & \text{if } v(p) = t\otimes \text{ or } v(p) = b\otimes \\ f\odot & \text{if } v(p) = f\odot \text{ or } v(p) = n\odot \\ f\oplus & \text{if } v(p) = f\oplus \text{ or } v(p) = n\oplus \\ f\ominus & \text{if } v(p) = f\ominus \text{ or } v(p) = n\ominus \\ f\otimes & \text{if } v(p) = f\otimes \text{ or } v(p) = n\otimes \end{cases} \qquad v'(\neg p) = \begin{cases} t\odot & \text{if } v(p) = f\odot \text{ or } v(p) = b\odot \\ t\oplus & \text{if } v(p) = f\ominus \text{ or } v(p) = b\ominus \\ t\ominus & \text{if } v(p) = f\oplus \text{ or } v(p) = b\oplus \\ t\otimes & \text{if } v(p) = f\otimes \text{ or } v(p) = b\otimes \\ f\odot & \text{if } v(p) = t\odot \text{ or } v(p) = n\odot \\ f\oplus & \text{if } v(p) = t\ominus \text{ or } v(p) = n\oplus \\ f\ominus & \text{if } v(p) = t\oplus \text{ or } v(p) = n\ominus \\ f\otimes & \text{if } v(p) = t\otimes \text{ or } v(p) = n\otimes \end{cases}$$

It can be proved by induction on the complexity of a formula that the clauses for obtaining v' extend to any formula in such a way that v' is a valuation over $\mathfrak{M}^{a\pm}_{sr\odot\oplus,glgp}$. Therefore $v'(\gamma) \notin \{f\odot, f\oplus\}$ for all $\gamma \in \Gamma$, but $v'(\delta) \in \{f\odot, f\oplus\}$ for all $\delta \in \Delta$, implying that $\Gamma \nvDash_{\mathfrak{M}^{a\pm}_{sr\odot\oplus,glgp}} \Delta$.

($\Leftarrow$) Suppose that $\Gamma \nvDash_{\mathfrak{M}^{a\pm}_{sr\odot\oplus,glgp}} \Delta$. Therefore there is a valuation v' over $\mathfrak{M}^{a\pm}_{s\odot\oplus,glgp}$ such that $v'(\gamma) \notin \{f\odot, f\oplus\}$ for all $\gamma \in \Gamma$, while $v'(\delta) \in \{f\odot, f\oplus\}$ for all $\delta \in \Delta$. Then construct a valuation v over $\mathsf{DM_4} \times \mathsf{IS_4}$ such that for all p:

$$v(p) = \begin{cases} t\odot & \text{if } v'(p) = t\odot \text{ and } v'(\neg p) = f\odot \\ t\oplus & \text{if } v'(p) = t\oplus \text{ and } v'(\neg p) = f\ominus \\ t\ominus & \text{if } v'(p) = t\ominus \text{ and } v'(\neg p) = f\oplus \\ t\otimes & \text{if } v'(p) = t\otimes \text{ and } v'(\neg p) = f\otimes \\ f\odot & \text{if } v'(p) = f\odot \text{ and } v'(\neg p) = t\odot \\ f\oplus & \text{if } v'(p) = f\oplus \text{ and } v'(\neg p) = t\ominus \\ f\ominus & \text{if } v'(p) = f\ominus \text{ and } v'(\neg p) = t\oplus \\ f\otimes & \text{if } v'(p) = f\otimes \text{ and } v'(\neg p) = t\otimes \end{cases} \qquad v(p) = \begin{cases} b\odot & \text{if } v'(p) = t\odot \text{ and } v'(\neg p) = t\odot \\ b\oplus & \text{if } v'(p) = t\oplus \text{ and } v'(\neg p) = t\ominus \\ b\ominus & \text{if } v'(p) = t\ominus \text{ and } v'(\neg p) = t\oplus \\ b\otimes & \text{if } v'(p) = t\otimes \text{ and } v'(\neg p) = t\otimes \\ n\odot & \text{if } v'(p) = f\odot \text{ and } v'(\neg p) = f\odot \\ n\oplus & \text{if } v'(p) = f\oplus \text{ and } v'(\neg p) = f\ominus \\ n\ominus & \text{if } v'(p) = f\ominus \text{ and } v'(\neg p) = f\oplus \\ n\otimes & \text{if } v'(p) = f\otimes \text{ and } v'(\neg p) = f\otimes \end{cases}$$

It can be proved by induction on the complexity of a formula that the clauses for obtaining v extend to any formula in such a way that v is a valuation over $\mathfrak{M}^{\mathbf{BD}}_{sr\oplus\odot}$. By this, $v(\gamma) \notin \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\gamma \in \Gamma$, while $v(\delta) \in \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\delta \in \Delta$. Hence, $\Gamma \nvDash_{\mathfrak{M}^{\mathbf{BD}}_{sr\oplus\odot}} \Delta$.

□

The second kind of non-deterministic structures to be discussed are defined over $\mathsf{B_2} \times \mathsf{IS_2}$, $\mathsf{K_3} \times \mathsf{IS_2}$ or $\mathsf{DM_4} \times \mathsf{IS_2}$ by replacing the negation function with one in Figure 4.3.

Definition 4.4.4.

- Let $\mathfrak{M}^{t}_{s\odot}$, $\mathfrak{M}^{t}_{sr\odot}$ be the Nmatrices and let $\boldsymbol{\mathfrak{M}}^{t}_{p\odot}$ and $\boldsymbol{\mathfrak{M}}^{t}_{pr\odot}$ be the class of Nmatrices, obtained by replacing the negation function of $\mathfrak{M}^{\mathbf{CL}}_{s\odot}$, $\mathfrak{M}^{\mathbf{CL}}_{sr\odot}$, $\boldsymbol{\mathfrak{M}}^{\mathbf{CL}}_{p\odot}$ and $\boldsymbol{\mathfrak{M}}^{\mathbf{CL}}_{pr\odot}$, respectively, with $\neg^{t}$.
- Let $\mathfrak{M}^{tLP}_{s\odot}$, $\mathfrak{M}^{tLP}_{sr\odot}$ be the Nmatrices and let $\boldsymbol{\mathfrak{M}}^{tLP}_{p\odot}$ and $\boldsymbol{\mathfrak{M}}^{tLP}_{pr\odot}$ be the classes of Nmatrices, obtained by replacing the negation function of $\mathfrak{M}^{\mathbf{LP}}_{s\odot}$, $\mathfrak{M}^{\mathbf{LP}}_{sr\odot}$, $\boldsymbol{\mathfrak{M}}^{\mathbf{LP}}_{p\odot}$ and $\boldsymbol{\mathfrak{M}}^{\mathbf{LP}}_{pr\odot}$, respectively, with $\neg^{t}_{gl}$.
- Let $\mathfrak{M}^{tK_3}_{s\odot}$, $\mathfrak{M}^{tK_3}_{sr\odot}$ be the Nmatrices and let $\boldsymbol{\mathfrak{M}}^{tK_3}_{p\odot}$ and $\boldsymbol{\mathfrak{M}}^{tK_3}_{pr\odot}$ be the classes of Nmatrices, obtained by replacing the negation function of $\mathfrak{M}^{\mathbf{K_3}}_{s\odot}$, $\mathfrak{M}^{\mathbf{K_3}}_{sr\odot}$, $\boldsymbol{\mathfrak{M}}^{\mathbf{K_3}}_{p\odot}$ and $\boldsymbol{\mathfrak{M}}^{\mathbf{K_3}}_{pr\odot}$, respectively, with $\neg^{t}_{gp}$.
- Let $\mathfrak{M}^{t}_{s\odot,glgp}$, $\mathfrak{M}^{t}_{sr\odot,glgp}$ be the Nmatrices and let $\boldsymbol{\mathfrak{M}}^{t}_{p\odot,glgp}$ and $\boldsymbol{\mathfrak{M}}^{t}_{pr\odot,glgp}$ be the classes of Nmatrices, obtained by replacing the negation function of $\mathfrak{M}^{\mathbf{BD}}_{s\odot}$, $\mathfrak{M}^{\mathbf{BD}}_{sr\odot}$, $\boldsymbol{\mathfrak{M}}^{\mathbf{BD}}_{p\odot}$ and $\boldsymbol{\mathfrak{M}}^{\mathbf{BD}}_{pr\odot}$, respectively, with $\neg^{t}_{glgp}$.

With these definitions, the following equalities can be proved.

$$\mathbf{CL^{wr\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{t}}_{\mathbf{s}\odot}} \quad \mathbf{CL^{wl\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{t}}_{\mathbf{sr}\odot}} \quad \mathbf{CL^{pr\pm}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{\mathbf{t}}_{\mathbf{p}\odot}} \quad \mathbf{CL^{pl\pm}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{\mathbf{t}}_{\mathbf{pr}\odot}}$$

$$\mathbf{LP}^{wr\pm} = \mathbf{L}_{\mathfrak{M}^{tLP}_{s\odot}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{tLP}_{p\odot}} \quad \mathbf{K_3}^{\mathbf{wr}\pm} = \mathbf{L}_{\mathfrak{M}^{tK_3}_{s\odot}}$$

$$\mathbf{K_3}^{\mathbf{pr}\pm} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{tK_3}_{p\odot}} \quad \mathbf{AC}_{\mathtt{fde}} = \mathbf{L}_{\mathfrak{M}^{t}_{s\odot,glgp}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{t}_{p\odot,glgp}}$$

$$\mathbf{LP^{wl\pm}} = \mathbf{L}_{\mathfrak{M}^{tLP}_{sr\odot}} \quad \mathbf{K_3}^{\mathbf{wl}\pm} = \mathbf{L}_{\mathfrak{M}^{tK_3}_{sr\odot}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{tK_3}_{pr\odot}}$$

$$\mathbf{LP^{pl\pm}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{tLP}_{pr\odot}} \quad \mathbf{BD^{wl\pm}} = \mathbf{L}_{\mathfrak{M}^{t}_{sr\odot,glgp}} = \mathbf{L}_{\boldsymbol{\mathfrak{M}}^{t}_{pr\odot,glgp}}$$

As before, to simplify the exposition, only the equality for $\mathbf{BD^{wl\pm}}$ is proved since the others employ the same constructions—also employed in Theorem Fact 4.2 in Szmuc and Zirattu, 2025—or can be obtained through restrictions thereof.

Theorem 4.4.2. $\Gamma \vDash_{\mathfrak{M}^t_{sr\odot,glgp}} \Delta$ *if and only if* $\Gamma \vdash_{\mathbf{BD^{wl\pm}}} \Delta$

Proof. As shown in section 4.3, $\mathbf{BD^{wl\pm}} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}}$. Thus, it is enough to prove that $\Gamma \vDash_{\mathfrak{M}^t_{sr\odot,glgp}} \Delta$ if and only if $\Gamma \vDash_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}} \Delta$. Both directions are proved by contraposition.

(⇒) Suppose that $\Gamma \nvDash_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}} \Delta$. So there is $v \in \mathrm{Hom}_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}}$ such that $v(\gamma) \notin \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\gamma \in \Gamma$, but $v(\delta) \in \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\delta \in \Delta$. Then construct a valuation v' over $\mathsf{DM_4} \times \mathsf{IS_2}$ such that for all literals p, $\neg p$:

$$v'(p) = \begin{cases} t\odot & \text{if } v(p) = t\odot \text{ or } v(p) = t\oplus \\ t\otimes & \text{if } v(p) = t\otimes \text{ or } v(p) = t\ominus \\ b\odot & \text{if } v(p) = b\odot \text{ or } v(p) = b\oplus \\ b\otimes & \text{if } v(p) = b\otimes \text{ or } v(p) = b\ominus \\ n\odot & \text{if } v(p) = n\odot \text{ or } v(p) = n\oplus \\ n\otimes & \text{if } v(p) = n\otimes \text{ or } v(p) = n\ominus \\ f\odot & \text{if } v(p) = f\odot \text{ or } v(p) = f\oplus \\ f\otimes & \text{if } v(p) = f\otimes \text{ or } v(p) = f\ominus \end{cases} \qquad v'(\neg p) = \begin{cases} t\odot & \text{if } v(p) = f\odot \text{ or } v(p) = f\ominus \\ t\otimes & \text{if } v(p) = f\otimes \text{ or } v(p) = f\oplus \\ b\odot & \text{if } v(p) = b\odot \text{ or } v(p) = b\ominus \\ b\otimes & \text{if } v(p) = b\otimes \text{ or } v(p) = b\oplus \\ n\odot & \text{if } v(p) = n\odot \text{ or } v(p) = n\ominus \\ n\otimes & \text{if } v(p) = n\otimes \text{ or } v(p) = n\oplus \\ f\odot & \text{if } v(p) = t\odot \text{ or } v(p) = t\ominus \\ f\otimes & \text{if } v(p) = t\otimes \text{ or } v(p) = t\oplus \end{cases}$$

It can be proved by induction on the complexity of a formula that the clauses for the construction of v' extend to complex formulas so that v' is a valuation over $\mathfrak{M}^t_{sr\odot,glgp}$. Therefore, $v'(\gamma) \notin \{f\odot, n\odot\}$ for all $\gamma \in \Gamma$, but $v'(\delta) \in \{f\odot, n\odot\}$ for all $\delta \in \Delta$, which implies that $\Gamma \nvDash_{\mathfrak{M}^t_{sr\odot,glgp}} \Delta$.

(⇐) Suppose that $\Gamma \nvDash_{\mathfrak{M}^t_{sr\odot,glgp}} \Delta$. So there is a valuation v' over $\mathfrak{M}^t_{sr\odot,glgp}$ such that $v'(\gamma) \notin \{f\odot, n\odot\}$ for all $\gamma \in \Gamma$, but $v'(\delta) \in \{f\odot, n\odot\}$ for all $\delta \in \Delta$. Then construct a valuation v over $\mathsf{DM_4} \times \mathsf{IS_4}$ such that for all p:

$$v(p) = \begin{cases} t\odot & \text{if } v'(p) = t\odot \text{ and } v'(\neg p) = f\odot \\ t\oplus & \text{if } v'(p) = t\odot \text{ and } v'(\neg p) = f\otimes \\ t\ominus & \text{if } v'(p) = t\otimes \text{ and } v'(\neg p) = f\odot \\ t\otimes & \text{if } v'(p) = t\otimes \text{ and } v'(\neg p) = f\otimes \\ f\odot & \text{if } v'(p) = f\odot \text{ and } v'(\neg p) = t\odot \\ f\oplus & \text{if } v'(p) = f\odot \text{ and } v'(\neg p) = t\otimes \\ f\ominus & \text{if } v'(p) = f\otimes \text{ and } v'(\neg p) = t\odot \\ f\otimes & \text{if } v'(p) = f\otimes \text{ and } v'(\neg p) = t\otimes \end{cases} \quad v(p) = \begin{cases} b\odot & \text{if } v'(p) = b\odot \text{ and } v'(\neg p) = b\odot \\ b\oplus & \text{if } v'(p) = b\odot \text{ and } v'(\neg p) = b\otimes \\ b\ominus & \text{if } v'(p) = b\otimes \text{ and } v'(\neg p) = b\odot \\ b\otimes & \text{if } v'(p) = b\otimes \text{ and } v'(\neg p) = b\otimes \\ n\odot & \text{if } v'(p) = n\odot \text{ and } v'(\neg p) = n\odot \\ n\oplus & \text{if } v'(p) = n\odot \text{ and } v'(\neg p) = n\otimes \\ n\ominus & \text{if } v'(p) = n\otimes \text{ and } v'(\neg p) = n\odot \\ n\otimes & \text{if } v'(p) = n\otimes \text{ and } u(\neg p) = n\otimes \end{cases}$$

It can be proved by induction on the complexity of a formula that the clauses for the construction of v extend to complex formulas so that v is a valuation over $\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}$. Therefore, $v(\gamma) \notin \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\gamma \in \Gamma$, but $v(\delta) \in \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\delta \in \Delta$, implying that $\Gamma \nvDash_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}} \Delta$.

□

The third and last family of non-deterministic structures is obtained as a combination of non-determinism on both sides of the two-address construction, employing the negation functions in Figure 4.4.

Definition 4.4.5.

- Let $\mathfrak{M}^{at}_{s\odot,gl}$, $\mathfrak{M}^{at}_{s\odot,gp}$ and $\mathfrak{M}^{at}_{s\odot,glgp}$ be the Nmatrices obtained replacing the negation function of $\mathfrak{M}^{\mathbf{CL}}_{s\odot}$ with $\neg^{at}_{\mathrm{gl}}$, $\neg^{at}_{\mathrm{gp}}$ and $\neg^{at}_{\mathrm{glgp}}$, respectively.

- Let $\mathfrak{M}^{at}_{sr\odot,gl}$, $\mathfrak{M}^{at}_{sr\odot,gp}$ and $\mathfrak{M}^{at}_{sr\odot,glgp}$ be the Nmatrices obtained replacing the negation function of $\mathfrak{M}^{\mathbf{CL}}_{sr\odot}$ with $\neg^{at}_{\mathrm{gl}}$, $\neg^{at}_{\mathrm{gp}}$ and $\neg^{at}_{\mathrm{glgp}}$, respectively.

- Let $\mathfrak{M}^{at}_{p\odot,gl}$, $\mathfrak{M}^{at}_{p\odot,gp}$ and $\mathfrak{M}^{at}_{p\odot,glgp}$ be the classes of Nmatrices obtained replacing the negation function of $\mathfrak{M}^{\mathbf{CL}}_{p\odot}$ with $\neg^{at}_{\mathrm{gl}}$, $\neg^{at}_{\mathrm{gp}}$ and $\neg^{at}_{\mathrm{glgp}}$, respectively.

- Let $\mathfrak{M}^{at}_{pr\odot,gl}$, $\mathfrak{M}^{at}_{pr\odot,gp}$ and $\mathfrak{M}^{at}_{pr\odot,glgp}$ be the classes of Nmatrices obtained replacing the negation function of $\mathfrak{M}^{\mathbf{CL}}_{pr\odot}$ with $\neg^{at}_{\mathrm{gl}}$, $\neg^{at}_{\mathrm{gp}}$ and $\neg^{at}_{\mathrm{glgp}}$, respectively.

The consequence relations induced by these Nmatrices stand in relation to some of the logics already considered above as described by the following equalities.

$$\mathbf{LP}^{wr\pm} = \mathbf{L}_{\mathfrak{M}^{at}_{s\odot,gl}} = \mathbf{L}_{\mathfrak{M}^{at}_{p\odot,gl}} \qquad \mathbf{K_3}^{\mathbf{wr}\pm} = \mathbf{L}_{\mathfrak{M}^{at}_{s\odot,gp}}$$

$$\mathbf{K_3}^{\mathbf{pr}\pm} = \mathbf{L}_{\mathfrak{M}^{at}_{p\odot,gp}} \qquad \mathbf{AC}_{\mathtt{fde}} = \mathbf{L}_{\mathfrak{M}^{at}_{s\odot,glgp}} = \mathbf{L}_{\mathfrak{M}^{at}_{p\odot,glgp}}$$

$$\mathbf{LP}^{\mathbf{wl}\pm} = \mathbf{L}_{\mathfrak{M}^{at}_{sr\odot,gl}} \qquad \mathbf{K_3}^{\mathbf{wl}\pm} = \mathbf{L}_{\mathfrak{M}^{at}_{sr\odot,gp}} = \mathbf{L}_{\mathfrak{M}^{at}_{pr\odot,gp}}$$

$$\mathbf{LP}^{\mathbf{pl}\pm} = \mathbf{L}_{\mathfrak{M}^{at}_{pr\odot,gl}} \qquad \mathbf{BD}^{\mathbf{wl}\pm} = \mathbf{L}_{\mathfrak{M}^{at}_{sr\odot,glgp}} = \mathbf{L}_{\mathfrak{M}^{at}_{pr\odot,glgp}}$$

As done previously, to keep the presentation concise, the proof is given only for the equality pertaining $\mathbf{BD}^{\mathbf{wl}\pm}$ as convenient restrictions allow to cover all the others. The following result rests on the construction employed in Fact 4.3 in Szmuc and Zirattu, 2025, which constitutes the semantic presentation of $\mathbf{AC}_{\mathtt{fde}}$ with the least number of values—*i.e.*, four values, with respect to the 7-, 9-, and 16-valued semantics for the same logic introduced in Randriamahazaka, 2022, Ferguson, 2017 and Fine, 2016a, respectively.

Theorem 4.4.3. $\Gamma \vDash_{\mathfrak{M}^{at}_{sr\odot,glgp}} \Delta$ *if and only if* $\Gamma \vdash_{\mathbf{BD}^{\mathbf{wl}\pm}} \Delta$

Proof. Also in this case, since $\mathbf{BD}^{\mathbf{wl}\pm} = \mathbf{L}_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}}$ as shown in section 4.3, it is sufficient to prove that $\Gamma \vDash_{\mathfrak{M}^{at}_{sr\odot,glgp}} \Delta$ if and only if $\Gamma \vDash_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}} \Delta$. Both directions are proved by contraposition.

($\Rightarrow$) Suppose that $\Gamma \nvDash_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}} \Delta$. Thus, there is a valuation $v \in \mathrm{Hom}_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}}$ such that $v(\gamma) \notin \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\gamma \in \Gamma$, while $v(\delta) \in \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\delta \in \Delta$. Then construct a valuation v' over $\mathsf{B}_2 \times \mathsf{IS}_2$ such that for all literals $p, \neg p$:

$$v'(p) = \begin{cases} t\odot & \text{if } v(p) = t\odot \text{ or } v(p) = t\oplus \text{ or } v(p) = b\odot \text{ or } v(p) = b\oplus \\ t\otimes & \text{if } v(p) = t\otimes \text{ or } v(p) = t\ominus \text{ or } v(p) = b\otimes \text{ or } v(p) = b\ominus \\ f\odot & \text{if } v(p) = f\odot \text{ or } v(p) = f\oplus \text{ or } v(p) = n\otimes \text{ or } v(p) = n\oplus \\ f\otimes & \text{if } v(p) = f\otimes \text{ or } v(p) = f\ominus \text{ or } v(p) = n\otimes \text{ or } v(p) = n\ominus \end{cases}$$

$$v'(\neg p) = \begin{cases} t\odot & \text{if } v(p) = f\odot \text{ or } v(p) = f\ominus \text{ or } v(p) = b\odot \text{ or } v(p) = b\ominus \\ t\otimes & \text{if } v(p) = f\otimes \text{ or } v(p) = f\oplus \text{ or } v(p) = b\otimes \text{ or } v(p) = b\oplus \\ f\odot & \text{if } v(p) = t\odot \text{ or } v(p) = t\ominus \text{ or } v(p) = n\odot \text{ or } v(p) = n\ominus \\ f\otimes & \text{if } v(p) = t\otimes \text{ or } v(p) = t\oplus \text{ or } v(p) = n\otimes \text{ or } v(p) = n\oplus \end{cases}$$

It can be proved by induction on the complexity of a formula that the clauses for the construction of v' extend to complex formulas so that v' is a valuation over $\mathfrak{M}^{at}_{sr\odot,glgp}$. By this, $v'(\gamma) \neq f\odot$ for all $\gamma \in \Gamma$, but $v'(\varphi) = f\odot$ for all $\delta \in \Delta$. Hence $\Gamma \nvDash_{\mathfrak{M}^{at}_{sr\odot,glgp}} \Delta$.

($\Leftarrow$) Suppose that $\Gamma \nvDash_{\mathfrak{M}^{at}_{sr\odot,glgp}} \Delta$. So there is a valuation v' over $\mathfrak{M}^{at}_{sr\odot,glgp}$ such that $v'(\gamma) \neq f\odot$ for all $\gamma \in \Gamma$, while $v'(\varphi) = f\odot$ for all $\delta \in \Delta$. Then construct a valuation over $\mathsf{DM}_4 \times \mathsf{IS}_4$ such that for any p:

$$v(p) = \begin{cases} t\odot & \text{if } v'(p) = t\odot \text{ and } v'(\neg p) = f\odot \\ t\oplus & \text{if } v'(p) = t\odot \text{ and } v'(\neg p) = f\otimes \\ t\ominus & \text{if } v'(p) = t\otimes \text{ and } v'(\neg p) = f\odot \\ t\otimes & \text{if } v'(p) = t\otimes \text{ and } v'(\neg p) = f\otimes \\ f\odot & \text{if } v'(p) = f\odot \text{ and } v'(\neg p) = t\odot \\ f\oplus & \text{if } v'(p) = f\odot \text{ and } v'(\neg p) = t\otimes \\ f\ominus & \text{if } v'(p) = f\otimes \text{ and } v'(\neg p) = t\odot \\ f\otimes & \text{if } v'(p) = f\otimes \text{ and } v'(\neg p) = t\otimes \end{cases} \qquad v(p) = \begin{cases} b\odot & \text{if } v'(p) = t\odot \text{ and } v'(\neg p) = t\odot \\ b\oplus & \text{if } v'(p) = t\odot \text{ and } v'(\neg p) = t\otimes \\ b\ominus & \text{if } v'(p) = t\otimes \text{ and } v'(\neg p) = t\odot \\ b\otimes & \text{if } v'(p) = t\otimes \text{ and } v'(\neg p) = t\otimes \\ n\odot & \text{if } v'(p) = f\odot \text{ and } v'(\neg p) = f\odot \\ n\oplus & \text{if } v'(p) = f\odot \text{ and } v'(\neg p) = f\otimes \\ n\ominus & \text{if } v'(p) = f\otimes \text{ and } v'(\neg p) = f\odot \\ n\otimes & \text{if } v'(p) = f\otimes \text{ and } v'(\neg p) = f\otimes \end{cases}$$

It can be proved by induction on the complexity of a formula that these clauses extend to any complex formula so that v is a valuation over $\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}$. Therefore, $v(\gamma) \notin \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\gamma \in \Gamma$, but $v(\delta) \in \{f\odot, f\oplus, n\odot, n\oplus\}$ for all $\delta \in \Delta$, which implies that $\Gamma \nvDash_{\mathfrak{M}^{\mathbf{BD}}_{sr\odot\oplus}} \Delta$.

□

The results presented so far show that non-determinism delivers a unified semantic framework for a wide range of (structural) logics of variable inclusion. This notably enriches the fruitfulness of the approach by Song et al., 2023 and can be favored for its parsimony as all of the systems discussed here were shown to be modeled employing the simplest two-address structure encountered, namely $\mathsf{B}_2 \times \mathsf{IS}_2$.

Moreover, the non-deterministic approach appears to open up several further avenues of application. These include, first, the already mentioned use of non-determinism in the degrees associated with the topical behavior of negation, and, more generally, the possibility of allowing non-deterministic behavior for other connectives as well. The latter, in particular, suggests a way of capturing sublogics of $\mathbf{AC}_{\texttt{fde}}$, such as the logic of factual equivalence studied in Correia, 2016, where, for example, the distributivity of conjunction over disjunction may fail. Finally, a further promising direction concerns the investigation of which variable inclusion logics can be obtained by relaxing some of the assumptions **(A1)**–**(A3)** and what would be the interpretation for their topical structure.

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