Basic Laws of Arithmetic.

Derived using concept-script

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Foreword V

In this book one finds theorems on which arithmetic is based, proven using signs that collectively I call concept-script.[a] The most important of these propositions,[b] some with an accompanying translation appended, are listed at the end. As may be seen, the investigation does not yet include the negative, rational, irrational and complex numbers, nor addition, multiplication, etc. Moreover, propositions about the cardinal numbers[c] are not yet present with the completeness initially planned. Missing, in particular, is the proposition that the number of objects falling under a concept is finite, if the number of objects falling under a superordinate[d] concept is finite. External reasons have made me postpone both this and the treatment of other numbers, and mathematical operations, to a sequel whose publication will depend on the reception of this first volume. What I have offered here may suffice to give an idea of my method. It might be thought that the propositions concerning the cardinal number Endlos[1] could have been omitted. To be sure, they are not needed for the foundation of arithmetic in its traditional extent; but their derivation is often easier than those of the corresponding propositions concerning finite cardinal numbers and can serve as preparation for the latter. Propositions also occur which are not

n.b.: alphabetical footnotes are translators' remarks, numerical footnotes are Frege's.

[a] We translate 'Begriffsschrift' as 'concept-script' when Frege is referring to the formal language or system, and leave the name of his 1879 book as '*Begriffsschrift*'. [in introduction: use of 'the concept-script' for system, leave out article if referred to language]

[b] '*Sätze*' — compare translators' introduction.

[c] '*Anzahlen*' — compare translators' introduction.

[d] '*übergeordnete*' — compare translators' introduction or Roy's appendix.

[1] Cardinal number of a countably infinite set.

about cardinal numbers but which are needed in proofs. They treat, for example, of following in a series, of single-valuedness[e] of relations, of composite and coupled relations,[f] of mapping by means of relations, and such like. These propositions could perhaps be allocated to an extended theory of combinations.

The proofs are contained solely in the sections entitled "Construction", while those headed "Analysis" are meant to assist the understanding by providing a preliminary and rough sketch of the proof. The proofs themselves contain no words but are carried out solely in my symbolism. They are presented as a series of formulae separated by continuous or | broken lines VI
or other signs. Each of these formulae is a complete proposition displaying all the conditions on which its validity[a] depends. This completeness, which does not tolerate any tacit addition of assumptions in thought, seems to me indispensable for the rigorous conduct of proof.

The progression from one proposition to the next proceeds by the rules which are listed in §48, and no transition is made that does not accord with these rules. How, and according to which rule, an inference is drawn is indicated by the sign standing between the formulae, while —— • —— marks the termination of a chain of inferences. For this purpose there have to be propositions which are not derived from others. Some of these are the basic laws listed in §47; others are definitions which are collected in a table at the end, together with a reference to their first occurrence. Time and again, the pursuit of this project will generate a need for definitions. Their governing principles are listed in §33. Definitions themselves are not creative, and in my view must not be; they merely introduce abbreviative notations (names), which could be dispensed with were it not for the insurmountable external difficulties that the resulting prolixity would cause.

The ideal of a rigorous scientific method for mathematics that I have striven to realise here, and which could be named after Euclid, can be characterised as follows. It cannot be required that everything be proven, as this is impossible; but it can be demanded that all propositions appealed to without proof are explicitly declared as such, so that it can be clearly recognised on what the whole structure rests. One must strive to reduce the number of these fundamental laws as far as possible by proving everything that is provable. Furthermore, and in this I go beyond Euclid, I demand that all modes of inference and consequence which are used be listed in advance. Otherwise compliance with the first demand cannot be secured. This ideal I believe I have now essentially achieved. Only in a few points could one impose even more rigorous demands. In order to attain more flexibility and

[e] *Eindeutigkeit* — Furth: 'many-oneness' — see introduction

[f] *zusammengesetzten und gekoppelten Beziehungen* — Furth: relative product and 'coupled' relations — note: Russell's notion of relative product, not in the modern sense — see Roy's appendix

[a] *Geltung* — essentially: truth — remark on it in introduction?

to avoid excessive length, I have allowed myself tacit use of permutation of subcomponents (conditions) and fusion of equal subcomponents, and have not reduced the modes of inference and consequence to a minimum. Anyone acquainted with my little book *Begriffsschrift* will gather from it how here too one could satisfy the strictest demands, but also that this would result in a considerable increase in extent.

Furthermore, I believe that the criticisms that can justifiably be made of | this book will pertain not to the rigour but rather only to the choice VII
of the course of proof and of the intermediate steps. Often several ways of conducting a proof are available; I have not tried to pursue them all and it is possible, indeed likely, that I have not always chosen the shortest. Let whoever has complaints on this score try to do better. Other matters will be disputable. Some might have preferred to increase the circle of permissible modes of inference and consequence, in order to achieve greater flexibility and brevity. However, one has to draw a line somewhere if one approves of my stated ideal at all; and wherever one does so, people could always say: it would have been better to allow even more modes of inference.

The gaplessness of the chains of inferences contrives to bring to light each axiom, each presupposition, hypothesis, or whatever one may want to call that on which a proof rests; and thus we gain a basis for an assessment of the epistemological nature of the proven law. Although it has already been announced many times that arithmetic is merely further developed logic, still this remains disputable as long as there occur transitions in the proofs which do not conform to acknowledged logical laws but rather seem to rest on intuitive knowledge.[a] Only when these transitions are analysed into simple logical steps can one be convinced that nothing but logic forms the basis. I have listed everything that can facilitate an assessment whether the chains of inferences are properly connected and the buttresses are solid. If anyone should believe that there is some fault, then he must be able to state precisely where, in his view, the error lies: with the basic laws, with the definitions, or with the rules or a specific application of them. If everything is considered to be in good order, one thereby knows precisely the grounds on which each individual theorem rests. As far as I can see, a dispute can arise only concerning my basic law of value-ranges[b] (V), which perhaps has not yet been explicitly formulated by logicians although one thinks in accordance with it if, e.g., one speaks of extensions of concepts. I take it to be purely logical. At any rate, the place is hereby marked where there has to be a decision.

My purpose demands some divergences from what is common in mathematics. Rigour of proof requires, as an inescapable consequence, an increase

[a] *anschauendes Erkennen* — see translators' introduction

[b] *Wertheverläufe* — a.k.a. "courses-of-values", add explanation, who translates how, etc. — in the introduction

in length. Whoever fails to keep an eye on this will indeed be surprised how cumbersome our proofs often are of propositions into which he would suppose he had an immediate insight, through a single act of cognition. This will be especially striking if one compares Mr Dedekind's essay, *Was sind und was sollen die Zahlen?*,[c] the most thorough study I have seen in recent times concerning the foundations of arithmetic. It pursues, in much | less space, the laws of arithmetic to a much higher level than here. This VIII
concision is achieved, of course, only because much is not in fact proven at all. Often, Mr Dedekind merely states that a proof follows from such and such propositions; he uses dots, as in "$\mathfrak{M}(A, B, C \ldots)$"; nowhere in his essay do we find a list of the logical or other laws he takes as basic; and even if it were there, one would have no chance to verify whether in fact no other laws were used, since, for this, the proofs would have to be not merely indicated but carried out gaplessly. Mr Dedekind too is of the opinion that the theory[a] of numbers is a part of logic; but his essay barely contributes to the confirmation of this opinion since his use of the expressions "system", "a thing belongs to a thing" are neither customary in logic nor reducible to something acknowledged as logical. I do not say this as a complaint; his procedure may have been the most appropriate for his purpose; I say this only to cast a brighter light upon my own intentions by contrast. The length of a proof should not be measured by the ell. It is easy to make a proof appear short on paper, by missing out many intermediate steps in the chain of inferences or by merely gesturing at them. Mostly, no doubt, one contents oneself with the obvious correctness of each step in a proof; and permissibly so, if the aim is merely to persuade of the truth of the proposition to be proven. However, if the aim is to convey insight into the nature of this obviousness, this procedure does not suffice; rather, one must write out all intermediate steps, so that the full light of awareness may fall upon them. Usually, mathematicians are merely concerned with the content of a proposition and that it be proven. Here the novelty is not the content of the proposition, but how its proof is conducted, on what foundations it rests. That this essentially different perspective also requires another kind of treatment must not put us off. When one of our propositions is proven in the usual manner, then a proposition that appears to be unnecessary for the proof will easily be overlooked. In a thorough examination of my proof given here, I believe, one will indeed realise its indispensability, unless an entirely different path is taken. Here and there one will perhaps also encounter conditions in our propositions that strike one as redundant at first, but which will prove to be necessary after all, or at least eliminable only by using a proposition to be proven for this specific purpose.

I here carry out a project that I already had in mind at the time of my

[c] *Translators' Note*: *The Nature and Meaning of Numbers* – [**add ref. in bib**, not here]
[a] *Lehre*

Begriffsschrift of the year 1879 and which I announced in my *Grundlagen der Arithmetik* of the year 1884.[1] By this | act I aim to confirm the conception IX
of cardinal number which I set forth in the latter book. The basis for my results is articulated there in §46, namely that a statement of number[a] contains a predication about a concept; and the exposition here rests upon it. If someone takes a different view, he should try to develop a sound and usable symbolic exposition on that basis; he will find that it will not work. No doubt in language the point is not so transparent; but if one pays close attention, one finds that even here there is mention of a concept, rather than of a group, an aggregate or suchlike, whenever a statement of number is made; and even if exceptions sometimes occur, the group or the aggregate is always determined by a concept, i.e., by the properties an object must have in order to belong to the group, while what unites the group into a group, or makes the system into a system, the relations of the members to each other, has absolutely no bearing on the cardinal number.

The reason why the implementation appears so late after the announcement is owing in part to internal changes within the concept-script which forced me to jettison a nearly completed handwritten work. This progress might be mentioned here briefly. The primitive signs used in my *Begriffsschrift* occur again here with one exception. Instead of the three parallel lines, I have chosen the usual equality-sign, for I have convinced myself that in arithmetic it possesses just that reference that I too want to designate. Thus, I use the word "equal" with the same reference as "coinciding with" or "identical with", and this is also how the equality-sign is actually used in arithmetic. The objection to this which might be raised would rest on insufficiently distinguishing between sign and what is designated. No doubt, in the equation '$2^2 = 2 + 2$' the sign on the left is different from the one on the right; but both designate or refer to the same number.[1] To the original primitive signs two have now been added: the smooth breathing, designating the value-range of a function, and a sign to play the role of the definite article in language. The introduction of value-ranges of a function is an essential step forward, thanks to which we achieve far greater flexibility. What previously had been derived signs can now be replaced by other, and indeed simpler, ones, although the definitions of single-valuedness of a relation, of following in a series, of mapping are essentially the same as those given partly in my *Begriffsschrift*, partly in my *Grundlagen der Arithmetik*. | Value-ranges, however, have a much more fundamental impor- X
tance; for I define cardinal numbers themselves as extensions of concepts,

[1]Compare the introduction and §§90 and 91 in my *Grundlagen der Arithmetik*, Breslau, Verlag von Wilhelm Koebner, 1884.

[a]*Zahlangabe*

[1]To be sure, I also say: the sense of the sign on the right is different from the one on the left; but the reference is the same. Compare my essay *Über Sinn und Bedeutung* in the *Zeitschrift f. Philos. u. philos. Kritik*, vol. 100, p. 25.

and extensions of concepts are value-ranges, according to my specification. So without the latter one would never be able to get by. The old primitive signs that re-occur outwardly unaltered, and whose algorithm has hardly changed, have however been provided with different explanations. What was formerly the content-stroke reappears as the horizontal. These are consequences of a deep-reaching development in my logical views. Previously I distinguished two components in that whose external form is a declarative sentence: 1) acknowledgement of truth, 2) the content, which is acknowledged as true. The content I called judgeable content. This now splits for me into what I call thought and what I call truth-value. This is a consequence of the distinction between the sense and the reference of a sign. In this instance, the thought is the sense of a proposition and the truth-value is its reference. In addition, there is the acknowledgment that the truth-value is the True. For I distinguish two truth-values: the True and the False. I have justified this in more detail in my above mentioned essay *Über Sinn und Bedeutung*. Here, it might merely be mentioned that only in this way can indirect speech be accounted for correctly. For in indirect speech, the thought, which is normally the sense of the proposition, becomes its reference. Only a thorough engagement with the present work can teach how much simpler and more precise everything is made by the introduction of the truth-values. These advantages alone already weigh heavily in favour of my conception, which at first sight might admittedly seem strange. Moreover, the nature of functions, in contrast to objects, is characterised more precisely than in my *Begriffsschrift*. Further, from this the distinction between functions of first and second level results. As elaborated in my lecture, *Function und Begriff*,[1] concepts and relations are functions as I extended the reference of the term, and so we also must distinguish concepts of first and second level and relations of equal and unequal level.

As one can see, the years since the publication of my *Begriffsschrift* and *Grundlagen* have not passed in vain: they have seen the work mature. But the very thing which I regard as essential progress serves, as I cannot conceal from myself, as a major obstruction to the dissemination and influence of this book. Moreover, what I regard as not the least of its virtues, strict gaplessness of the chains of inferences, will earn it, I fear, scant appreciation. I have departed further from traditional conceptions | and thereby XI
impressed on my views a paradoxical character. An expression, cropping up here and there as one leafs through the pages, will all too easily seem strange and provoke negative prejudice. I can myself gauge somewhat the resistance which my innovations will encounter, as I too had first to overcome something similar in order to make them. To be sure, I have arrived at them not arbitrarily and out of a craze for novelty, but was forced by the very subject matter itself.

[1] Jena, Verlag von Hermann Pohle.

With this, I arrive at a second reason for the delay: the despondency that at times overcame me as a result of the cool reception, or rather, the lack of reception, by mathematicians[1] of the writings mentioned above, and the unfavourable scientific currents against which my book will have to struggle. The first impression alone can only be off-putting: strange signs, pages of nothing but alien formulae. Thus sometimes I concerned myself with other subjects. Yet as time passed, I simply could not contain those results of my thinking, which seemed to me valuable, locked up in my desk; and work expended always called for further work if it was not to be in vain. Thus the subject matter kept me captive. In such a case, when the value of a book cannot be appreciated on a swift reading, the reviewer should step in to assist. But in general the remuneration will be too poor. The critic can never hope to be compensated in money for the effort that a thoroughgoing study of this book will demand. All that is left for me is to hope that someone may from the outset have sufficient confidence in the work to anticipate that his inner reward will be repayment enough, and will then publicise the results of a thorough examination. It is not that only a complimentary review could satisfy me; quite the contrary! I would always prefer a critical assault based on a thorough study to praise that indulges in generalities without engaging the heart of the matter. Now I would like to offer some hints to assist the labour of a reader approaching the book with these intentions.

In order to gain an initial rough idea of how I express thoughts with my signs, it will be helpful to look at some of the easier cases in the table of the more important theorems, to which a translation is appended. It will then be possible to surmise what is intended in further, similar examples which are not followed by a translation. Next, one should begin with the introduction and start to tackle the exposition of the concept-script. However, I advise first to make merely a summary overview of it | and not to dwell on particular XII
concerns. In order to meet all objections, some issues have had to be taken up which are not required for understanding concept-script propositions. I include in this the second half of §8 which starts on p. 12 with "If we now give the following explanation", and also the second half of §9, which starts on p. 15 with the words "If I say in general", together with the whole of §10. These passages should be omitted on a first reading. The same applies to §26 and §§28–32. By contrast, I wish to lay stress on the first half of §8, as well as §§12 and 13, as particularly important for the understanding. A more detailed reading should start with §34 and continue to the end. Occasionally, one will have to revisit §§ merely fleetingly read. The index at the end and the table of contents will facilitate this. The derivations in §§49–52 can be

[1]One searches in vain for my *Grundlagen der Arithmetik* in the *Jahrbuch über die Fortschritte der Mathematik*. Researchers in the same area, Mr Dedekind, Mr Otto Stolz, Mr von Helmholtz seem not to be acquainted with my works. Kronecker does not mention them in his essay on the concept of number either.

used as preparation for an understanding of the proofs themselves. Here, all modes of inference and nearly all of the applications of our basic laws already occur. When one has reached the end, one should reread the entire exposition of the concept-script with this as background, keeping in mind that those stipulations that will not be used later, and therefore appear unnecessary, serve to implement the principle that all correctly formed signs ought to refer to something — a principle that is essential for full rigour. In this way, I believe, the mistrust that my innovations may initially provoke will gradually disappear. The reader will recognise that my principles will in no case lead to consequences other than ones he must acknowledge as correct himself. Perhaps he will then admit that he had overestimated the labour, that, in fact, my gapless approach facilitates the understanding, once the barrier presented by the novelty of the signs is overcome. May I be fortunate enough to find such a reader or reviewer! For a review based on a superficial reading might easily do more harm than good.

Otherwise, of course, the prospects for my book are dim. In any case, we must give up on those mathematicians who, encountering logical expressions like "concept", "relation", "judgement", think: *metaphysica sunt, non leguntur!*[a] and also on those philosophers who, sighting a formula, cry out: *mathematica sunt, non leguntur!*[b] and the exceptions will be very few. Perhaps the number of mathematicians who care about the foundation of their science is not large in any case, and even those who do care often seem to be in a great hurry to leave the initial grounds[c] behind them. Moreover, I hardly dare hope that many of them will be convinced by my reasons for the painstaking rigour, and the lengthiness connected with it. Custom exerts great | power over the mind. If I compare arithmetic with a tree that high up XIII
unfolds in a multiplicity of methods and theorems, while the root stretches into the depths, then it seems to me that the growth of the root, at least in Germany, is weak. Even in the *Algebra der Logik* of Mr E. Schröder, a work one would want to count as pursuing this direction, upper growth soon dominates before any greater depth is attained, causing an upward bent and a ramification into methods and theorems.

Of further disadvantage for my book is a widespread tendency to accept as being[a] only what can be sensed. What cannot be perceived with the senses one tries to disown, or at least to ignore. Now the objects of arithmetic, the numbers, are imperceptible; how to come to terms with this? Very simple! Declare the number-signs to be the numbers. In the signs, one

[a] *Translators' Note*: [add explanation, source of the phrase, "graeca sunt..."].

[b] *Translators' Note*: see footnote above.

[c] *Anfangsgründe* — preliminaries? He might be referring to first chapters in maths textbooks. Check whether these are sometimes called "*Anfangsgründe*" at the time. — This recurs later: check it. — Maybe compare: Kant's *Metaphysische Anfangsgründe*: Metaphysical Foundations?

[a] *vorhanden* — see introduction

then has something visible; and this, of course, is the main thing. To be sure, the signs have properties completely different from the numbers; but so what? Just credit them with the desired properties by so-called definitions. To be sure, it is a puzzle how there can be a definition where there is no question of a connection between sign and what is designated. One kneads together sign and what is designated as indistinguishably as possible; depending on what is required, one can assert existence by appeal to their tangibility[1] or bring the true properties of the numbers to the foreground. On occasion, it seems that the number-signs are regarded like chess pieces, and the so-called definitions like rules of the game. In that case the sign designates nothing, but is rather the thing itself. One small detail is overlooked in all this, of course; namely that a thought is expressed by means of '$3^2 + 4^2 = 5^2$', whereas a configuration of chess pieces says nothing. When one is content with such superficialities, there is surely no basis for a deeper understanding.

Here it is crucial to get clear about what definition is and what it can achieve. Often one seems to credit it with a creative power, although in truth nothing takes place except to make something prominent by demarcation and designate it with a name. Just as the geographer does not create a sea when he draws borderlines and says: the part of the water surface bordered by these lines I will call Yellow Sea, so too the mathematician cannot properly create anything by his definitions. Moreover, a property which a thing just does not have cannot be magically attached to it by mere definition, except for the property of now being called by the name that one has given to it. That, however, an | egg-shaped figure, produced with ink on paper, XIV
may be endowed by definition with the property of resulting in One if added to One, I can only regard as scientific superstition. A lazy student could just as well be turned into a diligent one by means of definition alone. Unclarity develops easily here for want of the distinction between concept and object. If one says: "A square is a rectangle in which adjacent sides are equal", then one defines the concept *square* by stating what properties something must have in order to fall under it. I call these properties characteristic marks of the concept. Yet note that these characteristic marks of the concept are not its properties. The concept *square* is not a rectangle, it is only the objects that fall under this concept that are rectangles, just as the concept *black cloth* is neither black nor a cloth. Whether there are such objects is not immediately known on the basis of the definition. One wants to define the number Zero, for example, by saying: it is something which when added to One, results in One. Thus a concept is defined by stating what property an object must have in order to fall under it. This property, however, is

[1] Compare E. Heine, *Die Elemente der Functionslehre*, in Crelle's *Journal*, vol. 74, p. 173: "Concerning definitions, I take the purely formal standpoint in calling certain tangible signs numbers, so that the existence of these numbers is thus not in question."

not a property of the defined concept. Yet, as it seems, it is often imagined that something which added to One results in One is created by definition. What a great illusion! The defined concept does not possess this property, nor does the definition guarantee that the concept is instantiated. This first requires an investigation. Only when one has shown that there is one and only one object with the requisite property is one in a position to give this object the proper name "Zero". To create Zero is hence impossible. I have repeatedly spelt these things out but, seemingly, without success.[1]

A proper appreciation of the distinction I draw, between the characteristic mark of a concept and the property of an object, can scarcely be hoped for from the prevailing logic either,[2] for that seems to be contaminated with psychology through and through. If instead of the things themselves, one considers only their subjective images, their ideas,[a] then naturally all finer-grained, objective distinctions are lost and others appear in their place that are logically completely worthless. Thus I come to speak about the obstacle to the influence of my book on the logicians. It is the ruinous incursion of psychology into logic. Decisive for the treatment of this science is how the logical laws are conceived, and this in turn connects with how | one XV
understands the word "true". It is commonly granted that the logical laws are guidelines which thought should follow to arrive at the truth; but it is too easily forgotten. The ambiguity of the word "law" here is fatal. In one sense it says what is, in the other it prescribes what should be. Only in the latter sense can the logical laws be called laws of thought, in so far as they legislate[a] how one ought to think. Every law stating what is the case can be conceived as prescriptive, one should think in accordance with it, and in that sense it is accordingly a law of thought. This holds for geometrical and physical laws no less than for the logical. The latter better deserve the title "laws of thought" only if thereby it is supposed to be said that they are the most general laws, prescribing how to think wherever there is thinking at all. But the phrase "laws of thought" seduces one to form the opinion that these laws govern thinking in the same way that the laws of nature govern events in the external world. In that case they can be nothing other than psychological laws; for thinking is a mental process. And if logic had to do with psychological laws, it would be a part of psychology. And thus it is in fact conceived. These laws of thought may then be conceived as guidelines merely in the manner of stating a mean, similar to the way one can say how healthy digestion proceeds in humans, or how grammatically correct speech goes, or how one dresses fashionably. Then one can merely say: humans' taking to be true conforms on average to these laws, both at present and

[1]Mathematicians who prefer not to enter into the mazes of philosophy are requested to stop reading the foreword here.

[2]In the logic of Mr B. Erdmann I find no trace of this important distinction.

[a] *Vorstellungen* — see introduction

[a]*festsetzen* — **important: note** in introduction

wherever human beings are found; so, if one wants to stay in harmony with the mean, one had better follow suit. However, what is fashionable today will be out of fashion sometime, and is at present not fashionable amongst the Chinese; so, likewise, one can present psychological laws of thought as setting a standard only with restrictions. Indeed so, if logic deals with being taken to be true and not, rather, with being true! And that is what the psychological logicians conflate. Thus in the first volume of his *Logik*,[1] pp. 272 to 275, Mr B. Erdmann equates truth with general validity,[b] grounding this on general certainty[c] regarding the object judged, with this in turn on general consensus amongst those judging. And so, in the end, truth is reduced to being taken to be true by individuals. In opposition to this, I can only say: being true is different from being taken to be true, be it by one, be it by many, be it by all, and is in no way reducible to it. It is no contradiction that something | is true that is universally held to be false. By logical laws XVI
I do not understand psychological laws of taking to be true, but laws of being true. If it is true that I write this in my room on 13th July, 1893, while the wind is howling outside, then it remains true even if all humans should later hold it to be false. If being true is thus independent of anyone's acknowledgement, then the laws of being true are not psychological laws either but boundary stones which are anchored in an eternal ground, which our thinking may wash over but yet cannot displace. And because of this they set the standards for our thinking if it wants to attain the truth. Their relation to thinking is not like that of the grammatical laws to language, as if they were to give expression to the nature of our human thinking and vary with it. The conception of the logical laws according to Mr Erdmann is, of course, entirely different. He doubts their unconditional, eternal validity and wants to restrict them to our thinking as it is now (pp. 375ff). But "our thinking" can surely only mean the thinking of humanity up until now. Accordingly, the possibility remains open that human or other beings might be discovered who could execute judgements contradicting our logical laws. What if this were to happen? Mr Erdmann would say: so we see that those principles are not valid everywhere. Certainly! if they are to be psychological laws, they ought to be formulated in a way that makes explicit the genus of beings whose thinking is empirically governed by them. I would say: there are therefore beings who do not recognise certain truths immediately in the manner we do but are reliant, perhaps, on the more protracted way of induction. What, however, if beings were even found whose laws of thought directly contradicted ours, so that their application often led to opposite results? The psychological logician could only accept this and say: for them, those laws hold, for us these. I would say: here we have a hitherto

[1]Halle a. S., Max Niemeyer, 1892.

[b]*Allgemeingültigkeit* — Furth: general validity (he puts it in scare quotes) — see introduction

[c]*Allgemeingewissheit* — Furth: general certainty — see introduction

unknown kind of madness. He who thinks of logical laws as prescriptive of what ought to be thought, or as laws of what is true, rather than as natural laws concerning humans' taking to be true, will ask: Who is right? Whose laws of taking to be true are in accord with the laws of being true? The psychological logician cannot admit this question; for by so doing he would acknowledge laws of being true that were not psychological. Can the sense of the word "true" be subjected to a more damaging corruption than by the attempt to incorporate a relation to the judging subject! No-one will here object, surely, that the proposition, "I am hungry", could be true for one but false for another? The proposition, no doubt, but not the thought; for the word, "I", in the mouth of the other refers to a different person, | and XVII
the proposition, accordingly, expresses a different thought when it is uttered by him. All determinations of place, time, and so on, belong to the thought whose truth is at issue; being true itself is place- and timeless. How, then, is the principle of identity to be read? Is it like this: "It is impossible for humans in the year 1893 to acknowledge an object as being different from itself"? Or like this: "Every object is identical to itself"? The former law is about humans and contains a determination of time; in the latter, there is mention neither of humans nor of time. The latter is a law of being true; the former one of human taking to be true. Their content is entirely different, and they are independent of each other so that neither can be inferred from the other. This is why it is very confusing to designate both by the same name of the basic law of identity. Such confusions of fundamentally different things are to blame for the appalling unclarity which we find in the psychological logicians.

As to the question, why and with what right we acknowledge a logical law to be true, logic can respond only by reducing it to other logical laws. Where this is not possible, it can give no answer. Stepping outside logic, one can say: our nature and external circumstances force us to judge, and when we judge we cannot discard this law — of identity, for example — but have to acknowledge it if we do not want to lead our thinking into confusion and in the end abandon judgement altogether. I neither want to dispute nor to endorse this opinion, but merely note that what we have here is not a logical conclusion. What is offered here is not a ground of being true but of our taking to be true. And further: this impossibility, to which we are subject,[a] of rejecting the law does not prevent us from supposing beings who do so; but it does prevent us from supposing that such beings do so rightly; and it prevents us, moreover, from doubting whether it is we or they who are right. At least this is true of myself. If others dare in the same breath to both acknowledge a law and doubt it, then that seems to me to be an attempt to jump out of one's own skin against which I can only urgently warn. Whoever has once acknowledged a law of being true has thereby also acknowledged

[a] *die für uns besteht* — ?

a law that prescribes what ought to be judged, wherever, whenever and by whomsoever the judgement may be made.

Surveying the whole matter, it seems to me that different conceptions of truth lie at the source of the dispute. For me, it is something objective, independent of the judging subject, for psychological logicians, it is not. What Mr B. Erdmann calls "objective certainty" | is only a general acknowledgement by those who judge and cannot, accordingly, be independent of them but is liable to change with their mental nature. XVIII

We can capture this more generally still: I acknowledge a realm of the objective, non-actual, while the psychological logicians take the non-actual to be subjective without further ado. Yet it is utterly incomprehensible why something that has being[a] independently of the judging subject has to be actual, i.e., has to be capable of acting, directly or indirectly, upon the senses. No such connection between the concepts is to be found. One can even give examples to show the opposite. The number One, e.g., is not easily regarded as actual, unless one is a follower of J. S. Mill. On the other hand, it is impossible to credit each human with his own number One; for in that case we should first have to investigate to what extent the properties of these Ones agreed. And if someone said, "One times One is One", and another, "One times One is Two", then we could only register the difference and say: your One has that property, mine this. There could be no talk of a dispute about who is right or of an attempt to instruct; for there is no common object. Obviously this runs entirely contrary to the sense of the word "One" and the sense of the proposition "One times One is One". Since One, as the same for everybody, confronts everyone in the same way, it can no more be investigated by means of psychological observation than the Moon. Should there after all be ideas of the number One in individual minds, then these are still to be distinguished from the number One, just as ideas of the Moon are to be distinguished from the Moon itself. Since the psychological logicians fail to appreciate the possibility of the objective non-actual, they take concepts to be ideas and thereby assign them to psychology. But the true state of affairs asserts itself too forcefully for this to be accomplished easily. And hence a vacillation afflicts the use of the word "idea", so that sometimes it seems to refer to something which belongs to the mental life of the individual and which, in accordance with the psychological laws, amalgamates with other ideas, associates with them; while at other times, to something that confronts everyone in the same way, so that no bearer of ideas[b] is either mentioned or even presupposed. These two uses are incompatible; for the former, associations, amalgamations merely occur within the individual bearer of ideas[c] and merely occur at something

[a] *Bestand hat*

[b] *Translators' Note*: *Vorstellender* — Furth: possessor of the idea.

[c] *Translators' Note*: *Vorstellender* — Furth: mind

that is as private to the bearer of ideas as his joy or pain. It must never be forgotten that the ideas of different people, however similar they may be, what, by the way, we cannot ascertain precisely, nevertheless do not coincide but are to be distinguished. Everyone has his own ideas which cannot also belong to another. Here, of course, I understand "idea" in the psychological sense. The | vacillating use of the word causes unclarity and helps the XIX
psychological logicians conceal their weakness. When will this finally be put to an end! This way everything will eventually be dragged down into the realm of psychology; the boundary between the objective and the subjective is eroded further and further, and even actual objects are treated psychologically as ideas. For what is *actual* other than a predicate? And what are logical predicates other than ideas? Everything leads thus into idealism and therefore, as an unavoidable consequence, into solipsism. If everyone designated something different by the name "Moon", namely one of his ideas, much like he voices his pain with the exclamation, "ouch!", then of course a psychological viewpoint would be justified; but a dispute concerning the properties of the moon would be pointless: one could perfectly well assert of his moon the opposite of what another says of his with the same right. If we could apprehend nothing but what is internal to ourselves, then a conflict of opinion, a mutual understanding would be impossible since a common ground would be lacking, and such a common ground cannot be an idea in the sense of psychology. There would be no logic appointed to be arbiter in a conflict of opinions.

But lest I give the impression that I am tilting at windmills, let me illustrate this inescapable sinking into idealism with reference to a particular book. For this, I choose Mr B. Erdmann's above mentioned *Logik* as one of the most recent works of the psychological trend, one which might not be denied all significance. First, let us observe the following proposition[a] (I, p. 85):

> "Thus psychology teaches with certainty that the objects of memory and imagination, just as those of deranged hallucinatory and illusionary ideation[b], are of an ideal nature. ... Ideal, moreover, is the whole range of properly mathematical ideas,[c] from the number-series down to the objects of mechanics."

What a motley! So, the number Ten should stand on the same level as hallucinations! Here obviously the objective non-actual is being conflated with the subjective. Some objective things are actual, others not. *Actual* is only one of many predicates and is of no more concern to logic than, for instance,

[a] *Satz*
[b] *Vorstellens*
[c] *Vorstellungen*

the predicate *algebraic* as applied[d] to a curve. Naturally, this conflation ensnares Mr Erdmann in metaphysics, however much he strives to distance himself from it. I take it to be a sure sign of error should logic have to rely on metaphysics and psychology, sciences which themselves require logical principles. Where in that case is the real basic ground on which everything rests? Or is the situation like that of Münchhausen who pulled himself out of the bog by his own hair? I strongly doubt that this is possible | and surmise that Mr Erdmann remains enmired in the psychologico-metaphysical bog. XX

There is no real objectivity for Mr Erdmann; for everything is idea. Let us convince ourselves of this on the basis of his own statements. We read on p. 187 of the first volume:

> "As a relation between what is ideated,[a] a judgement presupposes at least two relata between which the relation holds. As a *predication* about what is ideated, it demands that one of these relata be determined as the object of which is predicated, the subject, ... the other as the object that is predicated, the predicate".

To begin with, we see here that both the subject of the predication and the predicate are designated as object or what is ideated. Here "what is ideated" could have been written instead of "object", for we read (I, p. 81): "For objects are what is ideated." And also conversely, everything ideated is meant to be object. On p. 38 one finds:

> "According to its origin, the ideated[b] divides into objects of sense perception and self-consciousness on the one hand, and into primitive and derived on the other."

But what has its source in sense perception or self-consciousness is of course mental in nature. The objects, what is ideated, and hence also subject and predicate, are thereby assigned to psychology. This is confirmed by the following passage (I, pp. 147 and 148):

> "It is the ideated or the idea in general.[c] For both are one and the same: the ideated is the idea, the idea what is ideated."

The word "idea" is indeed usually taken in a psychological sense; that this is also Mr Erdmann's use can be seen from the passages:

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[d] "*das Prädicat* algebraisch *von einer Curve ausgesagt*" — We used 'applied' in a similar context in vol. II, §56, p. 69 — see translators' introduction

[a] *Vorgestelltem*

[b] *das Vorgestellte*

[c] *die Vorstellung überhaupt*

> "Consciousness therefore is the genus of feeling, ideation,[d] wanting" (p. 35)

and

> "Ideation is composed of the ideas ... and the passages of ideas"[e] (p. 36).

After this we should not be surprised that an object comes into being in a psychological manner:

> "Insofar as a perception-mass ... presents the same as earlier stimuli and the excitations triggered by them, it *reproduces* the memory traces that originated from this same of the earlier stimuli and *amalgamates* with them into an object of the apperceived idea"[f] (I, p. 42).

On p. 43 it is then shown by way of an example how a steel engraving of Raphael's Sistine Madonna comes into being in a purely psychological way, without steel press, ink and paper. After | all this, no doubt can XXI
remain that the object about which a predication is made, the subject, is in Mr Erdmann's opinion taken to be an idea in the psychological sense of the word, as is the predicate, the object that is predicated. If this were right, then it could not be truthfully predicated of any subject that it was green, since there are no green ideas. Moreover, I could not predicate of any subject that it was independent of its being ideated[a] or of myself, the bearer of ideas, any more than my decisions can be independent of my wanting and of myself, the wanting subject; rather they would be destroyed with me, if I were destroyed. So there is no real objectivity for Mr Erdmann, which follows also from his taking the ideated or ideas in general, objects in the most general sense of the word, as highest genus ($\gamma\varepsilon\nu\iota\kappa\acute{\omega}\tau\alpha\tau o\nu$, genus summum) (p. 147). He is thus an idealist. If the idealists were consistent, they would regard the proposition, "Charlemagne conquered the Saxons",

[d] *das Vorstellen*

[e] "Ideation is composed of the ideas ⟨in which objects are given to us⟩ [*Translators' Note*: Following Furth, we have here and below completed the quotations from Erdmann's *Logik* with the passages left out by Frege. Here: "*in denen uns Gegenstände gegeben werden*".] and the passages of ideas ⟨by which these are remembered, newly combined, or predicatively analysed according to their associative relations⟩ [*Translators' Note*: Added from Erdmann: "*durch welche dieselben ihren Associationsbeziehungen gemäss erinnert, neu verknüpft oder prädikativ zerlegt werden*"]" — **Note to selves:** find a nice way of organising this footnote that doesn't lose any information and is user-friendly.

[f] "Insofar as a perception-mass ⟨e.g., the noise of a passing carriage⟩ [*Translators' Note*: Added from Erdmann: "*etwa das Geräusch eines vorrüberfahrenden Wagens*"] presents the same as earlier stimuli and the excitations triggered by them, it *reproduces* the memory traces that originated from this same of the earlier stimuli and *amalgamates* with them into an object of the apperceived idea" — **Note to selves:** as above

[a] *Vorgestelltwerden*

neither as true nor as false but as fiction, just as we are accustomed to understand, for example, the proposition, "Nessus carried Deïanira across the river Euenus"; for the proposition, "Nessus did not carry Deïanira across the river Euenus", could likewise only be true if the name "Nessus" had a bearer. It would probably not be straightforward to drive the idealists out of this point of view. But one does not have to tolerate that they corrupt the sense of the proposition in this way, as if I wanted to predicate something of my idea when I speak of Charlemagne; what I want is to designate a man who is independent of myself and my ideation and to predicate something of him. One can grant the idealists that the achievement of this intention is not entirely certain, that without wanting it, I perhaps lapse from truth into fiction. But this has no bearing on the sense. With the proposition, "This blade of grass is green", I predicate nothing of any idea of mine; I am not designating any of my ideas by means of the words, "this blade of grass"; and were I doing so, the proposition would be false. At this point a second falsification intrudes, namely, that my idea of the green is being predicated of my idea of this blade of grass. I repeat: there is in no way any mention of my ideas in this proposition; an entirely different sense is being smuggled in here. Incidentally, I do not understand at all how an idea can be predicated of something. It would equally be a falsification if one were to say that in the proposition, "The Moon is independent of me and my ideation",[b] my idea of independence of myself and my ideation is predicated of my idea of the Moon. This would be to surrender objectivity in the proper sense of the word, and to put something entirely different in its place. No doubt it is possible that, in making a judgement, such a play | of ideas should occur; but that is not the sense of the proposition. It may XXII also be observed that for one and the same proposition, and one and the same sense of the proposition, the play of ideas can be entirely different. Yet it is this logically irrelevant side-show which our logicians take as the proper object of their research.

How understandable it is that the nature of the subject matter recoils against sinking into idealism, and that Mr Erdmann does not want to admit that, for him, there is no real objectivity; but equally understandable is the futility of his endeavor. For if all subjects and predicates are ideas, and if all thinking is nothing but production, connection, change of ideas, then it is impossible to see how anything objective can ever be achieved. An indication of this futile resistance is the very use of the words "what is ideated" and "object" which at first apparently designate something objective, rather than an idea, but only apparently; for it becomes manifest that they refer to the same. To what purpose, then, this superfluity of expressions? This is not hard to guess. One may notice in addition that there is mention of

[b] *Vorstellen*

the object of an idea,[a] although the object is taken to be itself an idea. That would then be an idea of an idea. What relation between ideas might be designated by this? Unclear as this is, it is intelligible enough how, in the clash between the nature of the subject matter and idealism, such maelstroms can arise. Everywhere, we find the object of which I form an idea confused with this idea itself, only for their differences to come into prominence later. This conflict is manifest in the following proposition:

> "For an idea whose object is general is thus, as such, as an event of consciousness, no more general than an idea itself is real because its object is posited as real, or than an object experienced as sweet ... [b] is presented by ideas which themselves are sweet" (I, p. 86).

Here, the true state of affairs asserts itself with force. I could almost agree; but note that according to Erdmann's principles the object of an idea and the object which is presented by ideas are themselves ideas; and so we can see that all struggle is futile. Further, I ask to keep in mind the words "as such" that are similarly used on p. 83 in the following passage:

> "When actuality is predicated of an object, the real subject of a judgement is not the object or the ideated as such but is rather *the transcendent*, which is presupposed as the ground of being of the ideated, through which the ideated presents itself. Here, the transcendent should not be regarded as the un|knowable ... XXIII
> rather its transcendence is only to consist in its independence from being ideated.[a]"

Again, a vain attempt to haul oneself out of the bog! If we take these words seriously, then it is claimed that in this case the subject is not an idea. Yet if that is possible, then it cannot be seen why with other predicates, which express specific kinds of efficacy or actuality, the real subject must surely be an idea, e.g., as in the judgement "the earth is magnetic". So we would then arrive at the view that the real subject will be an idea in only a few judgements. However, once it is granted that it is not essential for either the subject or the predicate to be an idea, the rug is pulled out from under the whole psychological logic. All psychological considerations, which now swell our logic texts, thus prove to be pointless.

In fact, however, we probably should not take Mr Erdmann's notion of transcendence too seriously. I merely have to remind him of one of his statements (I, p. 148):

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[a] *Gegenstand der Vorstellung*
[b] **Note to selves:** fill in (in footnotes) here and at the end of the quote, as above?
[a] *Vorgestelltwerden*

> "Also subordinate to the highest genus is the *metaphysical* limit of our ideation, the transcendent",

and he is sunk; for the highest genus (*γενικώτατον*, *genus summum*) is, according to him, just the ideated, or the idea in general. Or might the word "transcendent" be used above in a different sense from here? In any case, one would suppose, the transcendent should be subordinate to the highest genus.

Let us dwell a moment longer on the expression "as such". I take the case where someone wants me to think that all objects are nothing but images on the retina of my eyes. Very well! I make no comment yet. But now he maintains that the tower is bigger than the window through which I take myself to be seeing it. To this, I would then say: either not both the tower and the window are retinal images in my eye, in which case the tower may be bigger than the window; or the tower and the window are, as you say, images on my retina, in which case the tower is not bigger but, rather, smaller than the window. At this point, he tries to relieve his embarrassment by resort to "as such", and says: the retinal image of the tower as such is, admittedly, not bigger than that of the window. Here I almost want to jump out of my skin and shout at him: well, in that case the retinal image of the tower is not at all bigger than that of the window; and if the tower were the retinal image of the tower and the window were the retinal image of the window, then the tower simply would not be bigger than the window, and if your logic teaches you otherwise, then it is good for nothing. This "as such" is an excellent invention of unclear writers who want to say neither yes nor no. However I do not tolerate such wavering between the two, | but rather ask: if actuality XXIV
is predicated of an object, is the real subject of the judgement the idea, yes or no? If not, then it arguably is the transcendent, which is presupposed as the ground of being of such an idea. But the transcendent is itself what is ideated or an idea. Thus we are driven to assume that the subject of the judgement is not the ideated transcendent, but rather the transcendent which is presupposed as the ground of being of this ideated transcendent. So we would have to go on forever; and no matter how far we were to go, we could never get past the subjective. Incidentally, the same game could also be initiated with the predicate, and not only with the predicate *actual* but just as well with, for example, *sweet*. We should then first say: if one predicates actuality or sweetness of an object, then the real predicate is not the ideated actuality or sweetness, but rather the transcendent which is presupposed as ground of the ideated. Yet we would not be able to come to rest with this, but would always be driven further. What can we learn from this? That psychological logic is on the wrong track when it conceives of the subject and predicate of judgements as ideas in the psychological sense, that psychological considerations are no more appropriate in logic than in astronomy and geology. If we ever want to get past the subjective, then

we have to think of cognition[a] as an activity that does not create what is cognised, but grasps what is already there. The image of grasping is well suited to elucidate the issue. When I grasp a pencil, many things take place in my body: stimulation of the nerves, changes in the tension and the pressure of muscles, tendons and bones, changes in the circulation of the blood. The sum of these processes, however, is not the pencil, nor do they create it. The latter has being[b] independently of these processes. It is essential to grasping that there is something which is grasped; the inner changes alone are not the grasping. Similarly, what we mentally apprehend has being[c] independently of this activity, of the ideas and their changes that are part of or accompany the apprehension; it is neither the sum of these processes nor is it created as part of our mental life.

Let us see further how subtler differences in the subject matter are smudged over by the psychological logicians. The point was already made in the case of characteristic mark and property. This is connected with the distinction I have emphasised between object and concept, as well as that between concepts of first and second level. Naturally, these differences are indiscernible by psychological logicians; for them everything is idea. For this reason, | the proper conception of those judgements which we express XXV
in English[a] by "there is"[b] also eludes them. This existence[c] is mixed up by Mr B. Erdmann (*Logik* I, p. 311) with actuality, which, as we saw, is also not clearly distinguished from objectivity. Of what are we in fact asserting that it is actual when we say, there are square roots of Four? Is it Two or −2? But neither the one nor the other is in any way named. And if I wanted to say that the number Two acted or was active or actual, then this would be false and quite different from what I want to say with the proposition, "There are square roots of Four". The confusion here before us is almost as bad as can be; since it does not involve concepts of the same level, but rather collapses a concept of the first level with a concept of the second. This is a hallmark of the obtuseness of psychological logic. Someone who has, generally, attained a more open point of view may wonder how such a mistake could be made by a professional logician; but before one can gauge the scale of such an error, one obviously has to recognise the distinction between concepts of first and second level in the first place, and psychological logic will presumably be incapable of that. The greatest barrier to this will be that the proponents are so exceedingly in awe of psychological depths, which however is nothing but psychological corruption of logic. And thus our thick logic books come about, bloated with unhealthy psychological lard,

[a] *Erkennen* — see introduction
[b] *besteht*
[c] *besteht*
[a] *Translators' Note*: In the original it says "*im Deutschen*".
[b] *es gibt*
[c] *Existenz*

concealing all finer details. A fruitful cooperation between mathematicians and logicians is thereby rendered impossible. While the mathematician defines objects, concepts and relations, the psychological logician is listening in on the coming and going of ideas, and in the end the mathematician's defining can only appear foolish to him, since it does not convey the nature of ideas. He looks into his psychological peep box[d] and says to the mathematician: I see nothing at all of what you are defining. And the latter can merely answer: no wonder! For it is not where you are looking for it.

This may suffice to put my logical standpoint into a clearer light by the contrast. The distance from psychological logic seems to me to be as wide as the sky, so much so that there is no prospect that my book will have an effect on it immediately. My impression is that the tree that I have planted has to heave an incredible load of stone to make space and light for itself. Still, I will not give up all hope that my book will eventually aid the overthrow of psychological logic. To make the proponents of the latter come to terms with my book, some acknowledgement from the mathematicians will not come amiss. And indeed, I believe that I can expect some support from this quarter, | since the mathematicians have in the end to make common XXVI
cause against the psychological logicians. As soon as the latter deign to engage with my book seriously, even if only in order to refute it, I shall take myself to have won. For the whole of part II is really a test of my logical convictions. It is from the outset unlikely that such a construction could be built on an insecure, defective basis. But if anyone has different convictions, let him try to build a similar construction on them and he will find, I believe, that it does not work, or at least that it does not work so well. And I could only acknowledge it as a refutation if someone indeed showed that a better, more enduring building can be erected on different basic convictions, or if someone proved to me that my basic principles lead to manifestly false conclusions. But no one will succeed in doing so. And so may this book, even if belatedly, contribute to a renaissance of logic.

Jena in July, 1893.

G. Frege

[d] *Translators' Note*: Frege uses the word '*Guckkasten*' here which is to be translated as 'peep box', rather than 'peep show' which has been used in translations so far (see e.g. Furth). Peep boxes are (often stereoscopic) devices into which pictures are inserted which are then looked at through one or two holes in the front. These boxes were very popular around the turn to the 20th century and are still to be found as children's toys today. 'Peep box' was apparently the term most commonly used at the time in Britain, and it is devoid of the risqué connotations that 'peep show' has. — The name of a popular brand of the toy is "View-Master".

Introduction 1

In my *Grundlagen der Arithmetik*[1] I aimed to make it plausible that arithmetic is a branch of logic and needs to rely neither on experience nor intuition as a basis for its proofs. In the present book this is now to be established by deduction of the simplest laws of cardinal number[a] by logical means alone. In order for this to be done convincingly, significantly higher demands have to be imposed on the conduct of proof than is usual in arithmetic.[2] We have to mark out in advance a few modes of inference and consequence, and no step is allowed to occur which is not in accordance with one of them. Thus, in the transition to a new judgement, one is no longer to be satisfied, as mathematicians up until now nearly always have been, with its obvious correctness,[b] rather it must be analysed into the simple logical steps of which it consists, and these are often not particularly few. In doing so, no presupposition must remain unnoticed; every required axiom has to be uncovered. For it is precisely the presuppositions that are made tacitly or without clear awareness that bar insight into the epistemological nature of a law.

For such an undertaking to succeed, the required concepts must, obviously, be sharply characterised. This applies especially to what mathematicians intend to designate by the word 'set'. *Dedekind*[3] seems to use the word 'system' with the same intention. But, the considerations which appeared four years earlier in my *Grundlagen* notwithstanding, no clear insight into the nature of the subject matter can be found in Dedekind, although he sometimes comes close to the core of things, as in this passage (p. 2):

> "Such a system S ... is completely determined if it is determined of every thing whether it is an element of S or not. Therefore the system S is the same as the system T, in signs $S = T$, if every element of S is also | an element of T and every element 2

[1] Breslau 1884. [Foundations of Arithmetic — **Add ref. to bib.**]

[a] *Anzahl* — see introduction

[2] Compare my *Grundlagen* §90.

[b] *als richtig einleuchten* — compare corresponding passage in the preface, p. VIII

[3] *Was sind und was sollen die Zahlen?* Braunschweig 1888. [*The Nature and Meaning of Numbers* (1963)]

> of T is also an element of S."

In contrast, other passages go astray, e.g., the following (pp. 1 and 2):

> "It very often happens that different things a, b, c ... considered for whatever reason under a common aspect, are joined together in the mind, so that they are said to form a *system* S."

Here the common aspect admittedly provides a presentiment of the truth; but this considering, this joining together in the mind, is no objective characteristic.[a] I ask: in whose mind? If they are joined together in one mind and not in another, do they still make up a system? What is supposed to be joined together in my mind surely must be in my mind. Do the things outside of me not compose systems? Is a system a subjective construction within a single mind? Is then the constellation Orion a system? And what are its elements? The stars, the molecules or the atoms? The following passage (p. 2) is noteworthy:

> "It is advantageous for the uniformity of expression to allow for the special case where a system S consists of a *single* (one and only one) element a; i.e., where the thing a is an element of S, but each thing distinct from a is not an element of S."

This is later (p. 3) so understood that every element s of a system S may itself be viewed as a system. Since in this case element and system coincide, it here becomes very perspicuous that according to *Dedekind*, it is in fact the elements that constitute[b] the system. In his *Vorlesungen über die Algebra der Logik*,[1] E. *Schröder* takes a step further than *Dedekind* by drawing attention to a connection which the latter has seemingly overlooked between his systems and concepts. In fact, what *Dedekind* actually means when he calls a system part of a system (p. 2) is the subordination of a concept under a concept, or the falling of an object under a concept, cases that he, like *Schröder*, fails to distinguish owing to a common error in their views; for *Schröder* too, at bottom, takes the elements to be what his *class* consists in. Thus, on his view, an empty class should not really occur any more than an empty system should on *Dedekind*'s view; yet the demand arising from the nature of the subject matter makes itself felt on both authors in different ways. *Dedekind* continues the passage quoted above:

> "In contrast, for specific reasons we here want wholly to exclude the empty system which contains no element, although it may be convenient for other investigations to invent such a fiction."

Such a fiction would thus be admissible; it is dispensed with only for certain reasons. *Schröder* ventures the fiction | of an empty class. So, it seems that 3

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[a] *Merkmal* — see introduction

[b] *Bestand ... ausmachen* — see introduction

[1] Leipzig 1890, p. 253. [Lectures on the Algebra of Logic]

both agree with many mathematicians that one may freely invent something that is not there, even something unthinkable; for if the system consists in its elements then the system will be abolished together with its elements. As concerns where the limits of such capricious fiction are, indeed whether there are any at all, little clarity and agreement may be found; yet the correctness of a proof may hinge on it. I believe I have settled this issue for all reasonable people, in my *Grundlagen der Arithmetik* (§92 and ff.) and in my lecture *Ueber formale Theorien der Arithmetik*.[1] *Schröder* invents his Zero and thereby entangles himself in great difficulties.[2] Accordingly, although there is lack of insight in both *Schröder* and *Dedekind*, the true situation makes itself felt every time a system has to be specified. *Dedekind* then cites the properties a thing must have in order to belong to the system, i.e., he defines a concept in virtue of its characteristic marks.[3] Now, if it is the characteristic marks that constitute[a] the concept, rather than the objects falling under it, then there are no problems and worries concerning an empty concept. Then, certainly, an object can never simultaneously be a concept; and a concept under which only one object falls must not be confused with it. So, in the end, the point will stand that a statement of number[b] contains a predication about a concept.[4] I have reduced cardinal number to the relation of equinumerosity and this to single-valued correlation.[c] Much the same applies to the word 'correlation' as to the word 'set'. Both are not uncommonly used in contemporary mathematics, usually without any deeper understanding of what one really wants to designate by them. If my thought, that arithmetic is a branch of pure logic, is correct then a purely logical expression for 'correlation' must be selected. I use 'relation' for this purpose. Concept and relation are the foundation stones on which I build my construction.

Yet even after the concepts are sharply circumscribed, it would be hard, almost impossible, to satisfy the demands necessarily imposed here on the conduct of proof without special auxiliary means. Such an auxiliary means is my concept-script, whose exposition will be my first task. The following may be noted in advance. It | will not always be possible to give a regular 4
definition of everything, simply because our ambition has to be to reduce

[1]Sitzungsberichte der Jenaischen Gesellschaft für Medicin und Naturwissenschaft, Jahrg. 1885, Sitzung vom 17. Juli. [*On Formal Theories of Arithmetic* — **add ref. to bib.**]

[2]Compare E. G. *Husserl* in Göttinger gel. Anzeigen, 1891, no. 7, p. 272, who however does not seem to untie the knot.

[3]On *concept, object, property, characteristic marks* compare my *Grundlagen* §38, 47, 53 and my essay *Ueber Begriff und Gegenstand* [*On concept and object* — **add ref. to bib.**] in Vierteljahresschrift für wissenschaftliche Philosophie, XVI, 2.

[a]*Bestand ... ausmachen* — see introduction

[b]*Zahlangabe* — see introduction

[4]§46 of my *Grundlagen*.

[c]*eindeutige Zuordnung* — see introduction

matters to what is logically simple, and this as such allows of no proper definition. In such a case, I have to make do with gesturing at what I mean. The important thing is that I be understood and therefore I will aim to unfold the subject matter gradually, rather than strive at the outset for full generality and final expression. Someone may perhaps wonder about the frequent use of quotation marks. It is by this means that I distinguish cases in which I speak of the sign itself from cases in which I speak of its reference. Pedantic though it may seem, I nevertheless take this to be necessary. It is remarkable how an imprecise manner of speaking and writing, perhaps originally used only for ease and brevity yet with full awareness of its imprecision, can finally addle the thinking after this awareness has disappeared. Thus people have managed to regard number-signs as numbers, the name as what is named, the mere auxiliary means as the object of arithmetic itself. Such experiences teach how necessary it is to place the highest demands on the accuracy of our manner of speaking and writing. And I have made every effort to meet them, at least in all cases where it seemed to me that something depends upon it.

I. Exposition of the concept-script 5

1. The primitive signs

Introduction to function, concept, relation[1]

§1. If the task is to give the original reference of the word 'function' as used in mathematics, then it is easy to slip into calling a function of x any expression that is formed from 'x' and certain specific numbers by means of the notations for sum, product, power, difference, etc. This is incorrect, since it presents the function as an *expression*, as a combination of signs, and not as what is designated by these. One will therefore be tempted to say 'reference of an expression' instead of 'expression'. Now, in this expression the letter 'x' occurs, which does not refer to a number like the sign '2' does but only indicates it indeterminately.[a] For different numerals put in place of 'x' we generally obtain different references. E.g., if we insert into the expression '$(2 + 3 . x^2) . x$' the numerals '0', '1', '2', '3' one after the other for x, then we obtain as reference the numbers 0, 5, 28, 87, respectively. None of these references can claim to be our function. The nature of the function reveals itself, rather, in the bond that it bestows on the numbers whose signs we put for 'x' and the numbers that then result as the reference of our expression; a bond which is manifested[b] by the course of the curve whose equation in rectangular coordinates is

$$\text{`}y = (2 + 3 . x^2) . x\text{'}$$

The nature of the *function* lies therefore in that part of the expression that is present without the 'x'. The expression of a *function* is *in need of completion*, *unsaturated*. The let|ter 'x' serves only to hold open places for a 6
numeral that is to complete the expression, and so marks the special kind of need for completion that constitutes the peculiar nature of the function just designated. In the sequel, instead of 'x' the letter 'ξ' will be used for this purpose.[1] This place-holding is to be understood is such a way that all places occupied by 'ξ' must always be filled just by the same, and never by different signs. I call these places *argument places*, that whose sign (name) takes this place in a particular case I call the *argument* of the function for this case. The function is completed by the argument; and that which results from this completion I call the *value* of the function for the argument.

[1] Compare my lecture on *Function und Begriff* (Jena 1891) and my paper on *Begriff und Gegenstand* in the Vierteljahrszeitschrift für wissensch. Phil. XVI, 2. My *Begriffsschrift* (Halle a. S. 1879) no longer corresponds entirely to my present standpoint; it is therefore to be consulted as an elucidation of what is presented here only with caution.

[a] *unbestimmt andeutet* — see introduction

[b] *sich anschaulich . . . darstellt* — see introduction re: '*Anschauung*'

[1] With this, however, nothing is meant to be stipulated for the concept-script. Rather, 'ξ' itself will never occur in the concept-script developments; I will only use it in the exposition of the concept-script and in elucidation.

Thus we obtain a name of the value of a function for an argument if we fill in the argument places of the name of the function with the name of the argument. So, e.g., '$(2+3\,.\,1^2)\,.\,1$' is a name of the number 5, composed of the function name '$(2+3\,.\,\xi^2)\,.\,\xi$' and '1'. So the argument is thus not to be considered part of the *function*, but serves rather to complete the, in itself, *unsaturated function.* When in what follows an expression like 'the function $\Phi(\xi)$' is used, it is always to be borne in mind that 'ξ' contributes to the designation of the function only insofar as it marks its argument places, and that the nature of the function would be unchanged if any other sign were put for 'ξ'.

§2. As means of generating functions, one has added the taking of limits in its various forms as infinite series, differential quotient, integral; and the word 'function' has come finally to be understood in such a general way that the connection between the value of a function and its argument may no longer be expressible by the signs of analysis, but only by words. A further extension has consisted in admitting complex numbers as arguments and hence also as function-values. In both these directions I have gone further. For on the one hand the signs of analysis were not always sufficient, and on the other hand not all of them were being used for the formation of function-names in that, e.g., '$\xi^2 = 4$' and '$\xi > 2$' were not accepted as names of functions, as I accepted them to be. But with this it is acknowledged at the same time that the range of function-values cannot remain restricted to numbers; for if I take the numbers 0, 1, 2, 3, one after the other, as the argument of the function, $\xi^2 = 4$, then I do not obtain numbers.

$$`0^2 = 4\text{'},\ `1^2 = 4\text{'},\ `2^2 = 4\text{'},\ `3^2 = 4\text{'}$$

are expressions of thoughts, some true, some false. I express it | like this: the 7
value of the function $\xi^2 = 4$ is either the *truth-value* of the true, or that of the false.[1] It is already clear from this that I do not want to assert anything yet when I simply write down an equation, but that I merely *designate* a truth-value; just as I assert nothing when I simply write down '2^2', but merely *designate* a number. I say: the names '$2^2 = 4$' and '$3 > 2$' *refer to* the same truth-value, which I call for short *the True.* Likewise, for me, '$3^2 = 4$' and '$1 > 2$' *refer to* the same truth-value, which I call for short *the False*, exactly as the name '2^2' *refers to* the number Four. Accordingly, I call the number Four the *reference* of '4' and '2^2', and I call the True the reference of '$3 > 2$'. I distinguish, however, the *reference* of a name from its *sense.* '2^2' and '$2 + 2$' do not have the same *sense*, and nor do '$2^2 = 4$' and '$2 + 2 = 4$' have the same *sense.* The sense of a name of a truth-value I call a *thought.* I say further that a name *expresses* its sense and *refers to* its reference. I *designate* with a name that what it refers to.

[1] I argued for this in more detail in my essay *Über Sinn und Bedeutung* in the Zeitschrift für Philosophie und philosophische Kritik, vol. 100.

The function $\xi^2 = 4$ can therefore only have two values, namely the True for the arguments 2 and -2 and the False for every other argument.

Moreover, the domain of what is admissible as an argument has to be expanded and extended to objects in general. *Objects* stand opposed to functions. Accordingly, I count as an *object* everything that is not a function, e.g., numbers, truth-values and the value-ranges introduced below. Thus, names of objects, the *proper names*, do not in themselves carry argument places; like the objects themselves, they are saturated.

§3. I use the words

> "the function $\Phi(\xi)$ has the same *value-range* as the function $\Psi(\xi)$"

throughout as co-referential with the words

> "the functions $\Phi(\xi)$ and $\Psi(\xi)$ always has the same value for the same argument."

This is the case with the functions $\xi^2 = 4$ and $3 \,.\, \xi^2 = 12$, at least if numbers are taken as arguments. We can, however, also think of the signs for square and multiplication as defined in such a way that the function

$$(\xi^2 = 4) = (3 \,.\, \xi^2 = 12)$$

has the True as value for every arbitrary argument. Here an expression of logic can also be used: "the concept *square | root of 4* has the same extension 8
as the concept *something such that the triple of its square is 12*". For such functions, whose value is always a truth-value, we can hence say 'extension of the concept' instead of 'value-range of the function'; and it seems appropriate simply to call a *concept* any function whose value is always a truth-value.

§4. So far only functions with a single argument have been talked about; but we can easily pass on to *functions with two arguments*. These stand *in need of double completion* insofar as a function with one argument is obtained after their completion by one argument has been effected. Only after yet another completion do we arrive at an object, and this object is then called the *value* of the function for the two arguments. Just as the letter 'ξ' served us in the case of functions with one argument, so here we use the letters 'ξ' and 'ζ' to indicate the double unsaturatedness of functions with two arguments, as in

$$`(\xi + \zeta)^2 + \zeta'.$$

By inserting e.g., '1' for 'ζ', we saturate the function in such a way that we have in $(\xi + 1)^2 + 1$ only a function with one argument. This use of the letters 'ξ' and 'ζ' must always be kept in view, whenever an expression like 'the function $\Psi(\xi, \zeta)$' occurs (cf. 2nd fn. in §1). I call ξ-*argument places* the places in which 'ξ' stands and ζ-*argument places* those in which 'ζ' stands. I say that the ξ-argument places are *related* to one another, and likewise the ζ-argument places, while I do not describe a ξ-argument place as *related* to a ζ-argument place.

The functions with two arguments, $\xi = \zeta$ and $\xi > \zeta$, always have a truth-value as value (at least if the signs '=' and '>' are explained in the appropriate way). Such functions we will suitably call relations. In the first relation, for example, 1 stands to 1, in general every object to itself; in the second, for example, 2 stands to 1. We say that an object Γ *stands in the relation* $\Psi(\xi, \zeta)$ *to* the object Δ if $\Psi(\Gamma, \Delta)$ is the True. Likewise we say that the object Δ *falls under* the concept $\Phi(\xi)$ if $\Phi(\Delta)$ is the True. It is here presupposed, of course, that the function $\Phi(\xi)$, and equally $\Psi(\xi, \zeta)$, always has a truth-value.[1]

| *Signs for Functions* 9

§5. Above it is already stated that within a mere equation no assertion is yet to be found; with '$2 + 3 = 5$' only a truth-value is designated, without its being said which one of the two it is. Moreover, if I wrote '$(2 + 3 = 5) = (2 = 2)$' and presupposed that one knows that $2 = 2$ is the True, even then I would not thereby have asserted that the sum of 2 and 3 is 5; rather I would only have designated the truth-value of: that '$2 + 3 = 5$' refers to the same as '$2 = 2$'. We are therefore in need of another special sign in order to be able to assert something as true. To this end, I let the sign '$\vdash$' precede the name of the truth-value, in such a way that, e.g., in

$$\text{`}\vdash 2^2 = 4\text{'}$$ [1]

it is asserted that the square of 2 is 4. I distinguish the *judgement* from the *thought* in such a way that I understand by a *judgement* the acknowledgement of the truth of a *thought*. The concept-script representation of a judgement by means of the sign '$\vdash$' I call a *concept-script proposition* or *proposition*[a] for short. I regard '$\vdash$' as composed of the vertical stroke, which I call the *judgement-stroke*, and the horizontal stroke, which I now propose

[1] There is a difficulty here which can easily obscure the true state of affairs and thereby arouse suspicion concerning the correctness of my conception. If we compare the expression, 'the truth-value of: than Δ falls under the concept $\Phi(\xi)$' with '$\Phi(\Delta)$' then we see that the '$\Phi(\ \ )$' really corresponds to 'the truth-value of: that ( ) falls under the concept $\Phi(\xi)$', and not to 'the concept $\Phi(\xi)$'. So the latter words do not really designate a concept (in our sense), even though the linguistic form makes it look as if they do. On the predicament in which language here finds itself, cf. my essay *Ueber Begriff und Gegenstand.*

[1] Here, I often use notations for sum, product, power, in a provisional way, although they are not yet defined, in order to be able to form convenient examples and to facilitate the understanding through hints. It should be kept in mind, though, that nothing rests on the references of these notations.

[a] *Satz* — see introduction

to label simply the *horizontal.*[2] The horizontal will mostly occur conjoined with other signs, as it does here with the judgement-stroke, and will thereby be protected from confusion with the minus-sign. Where it does occur separately, it has to be made somewhat longer than the *minus-sign.* I regard it as a function name such that

$$— \Delta$$

is the True when Δ is the True, and is the False when Δ is not the True.[3] Accordingly,

$$— \xi$$

is a function whose value is always a truth-value, or a concept in terms of | 10
our stipulation. Under this concept falls the True and only the True. Thus

$$\text{‘}— 2^2 = 4\text{’}$$

refers to the same as ‘$2^2 = 4$’, namely the True. For, in order to dispense with brackets, I specify that everything standing right of the horizontal, occupying the argument place of the function $— \xi$, should be conceived of as a whole unless brackets prohibit this.

$$\text{‘}— 2^2 = 5\text{’}$$

refers to the False, and hence the same as ‘$2^2 = 5$’, whereas

$$\text{‘}— 2\text{’}$$

refers to the False, and hence something different from the number 2. If Δ is a truth-value, then $— \Delta$ is the same truth-value, with the result that

$$\Delta = (— \Delta)$$

is the True. The latter, however, is the False if Δ is not a truth-value. We can accordingly say that

$$\Delta = (— \Delta)$$

[2]Earlier I called it the *content-stroke*, when I combined under the expression ‘judgeable content’ that what I now have learnt to distinguish as truth-value and thought. Cf. my paper *Über Sinn und Bedeutung.*

[3]Evidently, the sign ‘Δ’ must not be without reference, but it has to refer to an object. Names without reference must not occur in concept-script. The stipulation is made such that under all circumstances ‘$— \Delta$’ refers to something, provided only that ‘Δ’ refers to something. Otherwise $— \xi$ would not be a concept with sharp boundaries, thus in our sense not be a concept at all. I here use the *capital Greek letters* as if they were names referring to something, without stating their reference. Proceeding within concept-script itself, they, just as ‘ξ’ and ‘ζ’, will not occur.

is the truth-value of: that Δ is a truth value.

Accordingly, the function $—\,\Phi(\xi)$ is a concept, and the function $—\,\Psi(\xi, \zeta)$ is a relation, irrespective of whether or not $\Phi(\xi)$ is a concept or $\Psi(\xi, \zeta)$ is a relation.

Of the two signs of which '$\vdash$' is composed, only the judgment-stroke contains the assertion.

§6. We do not need a specific sign to explain a truth-value as the False provided we have a sign by means of which every truth-value is transformed into its opposite, which is in any case indispensable. I now stipulate:

The value of the function

$$\top\,\xi$$

is to be the False for every argument for which the value of the function

$$—\,\xi$$

is the True; and it is to be the True for all other arguments.

We thus have in

$$\top\,\xi$$

a function whose value is always a truth-value: it is a concept under which all objects fall with the sole exception of the True. From this it follows that '$\top\,\Delta$' always refers to the same as '$\top\,(—\,\Delta)$', as '$—\,\top\,\Delta$', and as '$—\,\top\,(—\,\Delta)$'. We therefore regard '$\top$' as composed of the small vertical stroke, the *negation-stroke*, and the two parts of the horizontal stroke of which each can be regarded as a *horizontal* in our sense. The transition from '$\top\,(—\,\Delta)$' or from '$—\,\top\,\Delta$' to '$\top\,\Delta$', as well as that from '$—\,—\,\Delta$' to '$—\,\Delta$', I will call the *fusion* of horizontals.

| According to our stipulation, $\top\,2^2 = 5$ is the True; thus 11

$$\vdash\!\top\,2^2 = 5,$$

in words: $2^2 = 5$ is not the True; or: the square of 2 is not 5.

Thus also: $\vdash\!\top\,2$.

§7. We have already used the equality-sign rather casually to form examples but it is necessary to stipulate something more precise regarding it.

$$\text{`}\Gamma = \Delta\text{'}$$

refers to the True, if Γ is the same as Δ; in all other cases it is to refer to the False.

In order to dispense with brackets, I specify that everything standing to the left of the equality-sign up to the nearest horizontal, as a whole refers to the ξ-argument of the function $\xi = \zeta$, insofar as *brackets* do not prevent this; that everything standing to the right of the equality-sign up to the nearest equality-sign collectively refers to the ζ-argument of this function, insofar as *brackets* do not prevent this (compare p. 10).

§8. We considered in §3 the case where an equation such as

$$‘\Phi(x) = \Psi(x)’$$

always yields a name for the True, whatever proper name we might insert for ‘x’, provided only that this really refers to an object. We then have the generality of an equality, while in ‘$2^2 = 4$’ we merely have an equality. This difference manifests itself thus: in the former case we have a letter ‘x’ that only indicates indeterminately, while in ‘$2^2 = 4$’ every sign has a determinate reference.[a] In order to obtain an expression for generality, one might have the idea of defining:

> “Let us understand ‘$\Phi(x)$’ as the True, if the value of the function $\Phi(\xi)$ is the True for every argument; otherwise it shall refer to the False.”

Here, it would be presupposed, as in all our considerations of this kind, that ‘$\Phi(\xi)$’ always acquires a reference, if we replace ‘ξ’ by a name that refers to an object. Otherwise, I would not call $\Phi(\xi)$ a *function.* Accordingly, ‘$x \,.\, (x - 1) = x^2 - x$’[b] would refer to the True, at least if the notations for multiplication, subtraction and squaring were defined to apply also to objects that are not numbers, so as to allow the equation to hold generally. In contrast, ‘$x \,.\, (x-1) = x^2$’ would refer to the False, because we obtain the False as reference, if we insert ‘1’ for ‘x’, although we obtain the True if we insert ‘0’. But in this stipulation the scope of generality is not sufficiently demarcated. One would, e.g., be in doubt whether ‘$\top\, 2+3 \,.\, x = 5 \,.\, x$’ would have to be understood as the negation of a generality or as the generality
of a negation; more precisely, whether this | should refer to the truth-value 12
of: that not for every argument the value of the function $2 + 3 \,.\, \xi = 5 \,.\, \xi$ is the True, or whether it should refer to the truth-value of: that for every argument the value of the function $\top\, 2+3 \,.\, \xi = 5 \,.\, \xi$ is the True. In the first case ‘$\top\, 2 + 3 \,.\, x = 5 \,.\, x$’ would refer to the True, in the other the False. It must, however, be possible to express the generality of a negation, as well as the negation of a generality. I will express the former as follows:

$$‘\overset{\mathfrak{a}}{\smile}\!\top\, 2 + 3 \,.\, \mathfrak{a} = 5 \,.\, \mathfrak{a}’$$

and the negation of a generality thus:

$$‘\top\!\overset{\mathfrak{a}}{\smile}\, 2 + 3 \,.\, \mathfrak{a} = 5 \,.\, \mathfrak{a}’$$

and the generality itself thus:

---

[a] *Translators' Note*: Frege's use of words with the same stem, ‘*andeuten*’ (‘indicate’) and ‘*bedeuten*’ (‘refer’), is lost in the translation.

[b] *Translators' Note*: Typo, noted by Thiel: the original has ‘$x^2 - 1$’.

$$\text{‘}\overset{\mathfrak{a}}{\smile}\, 2 + 3 \,.\, \mathfrak{a} = 5 \,.\, \mathfrak{a}\text{’}$$

The latter would refer to the True, if for every argument the value of the function $2 + 3 \,.\, \xi = 5 \,.\, \xi$ were the True. Because this is not the case,

$$\overset{\mathfrak{a}}{\smile}\, 2 + 3 \,.\, \mathfrak{a} = 5 \,.\, \mathfrak{a}$$

is the False, and therefore

$$\top\!\overset{\mathfrak{a}}{\smile}\, 2 + 3 \,.\, \mathfrak{a} = 5 \,.\, \mathfrak{a}$$

is the True.

$$\overset{\mathfrak{a}}{\smile}\!\top\, 2 + 3 \,.\, \mathfrak{a} = 5 \,.\, \mathfrak{a}$$

is the False, since it is not the case that the value of the function $\top\, 2+3 \,.\, \xi = 5 \,.\, \xi$ is the True for every argument; because for the argument 1 it is the False. Accordingly,

$$\top\!\overset{\mathfrak{a}}{\smile}\!\top\, 2 + 3 \,.\, \mathfrak{a} = 5 \,.\, \mathfrak{a}$$

is the True and

$$\text{‘}\vdash\!\top\!\overset{\mathfrak{a}}{\smile}\!\top\, 2 + 3 \,.\, \mathfrak{a} = 5 \,.\, \mathfrak{a}\text{’}$$

says: *there is* at least one solution for the equation ‘$2+3 \,.\, x = 5 \,.\, x$’. Likewise:

$$\vdash\!\top\!\overset{\mathfrak{a}}{\smile}\!\top\, \mathfrak{a}^2 = 1;$$

in words: *there is* at least one square root of 1. One sees from this how ‘there is’ is rendered in concept-script.

If we now give the following explanation:

> let
>
> $$\text{‘}\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})\text{’}$$
>
> refer to the True if the value of the function $\Phi(\xi)$ is the True for every argument, and otherwise the False;

then this requires supplementation in that one needs to state more precisely which function $\Phi(\xi)$ is in each case. We will call it the *corresponding* function. For there could be doubts. $\Delta = \Delta$ is the value both of the function $\Delta = \xi$ and the value of the function $\xi = \xi$, in both cases for the argument Δ. So one might, starting from $\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$, want to take as the corresponding function $\xi = \mathfrak{a}$, $\mathfrak{a} = \xi$, or $\xi = \xi$. | With our use of the German letters, 13
however, we would in the first two cases not even have a *function* because ‘$\xi = \mathfrak{a}$’ and ‘$\mathfrak{a} = \xi$’ always remain without reference, whatever one may insert for ‘ξ’; because the German letter ‘$\mathfrak{a}$’ ought not occur without ‘$\overset{\mathfrak{a}}{\smile}$’ prefixed, except in ‘$\overset{\mathfrak{a}}{\smile}$’ itself. Here only $\xi = \xi$ can thus be considered as the corresponding function. It is not so easy in the case of an expression like

$$`\overset{\mathfrak{a}}{\smile}((\mathfrak{a} + \mathfrak{a} = 2 \,.\, \mathfrak{a}) = (\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}))'$$

If one were to proceed blindly one might think to have the corresponding function in

$$`(\xi + \xi = 2 \,.\, \xi) = (\overset{\xi}{\smile}\, \xi = \xi)'$$

We now want to say that '$\mathfrak{a}$' stands above a *concavity* in '$\overset{\mathfrak{a}}{\smile}$'. The place above the concavity is never an *argument place*; thus at least the '$\mathfrak{a}$' standing above the second concavity has to be preserved. But since '$\overset{\mathfrak{a}}{\smile}$' must always be followed by a combination of signs that contain '$\mathfrak{a}$', '$\mathfrak{a}$' must be preserved in at least one of the two places in '$\mathfrak{a} = \mathfrak{a}$'. Accordingly one could surmise that the following functions were the corresponding ones

$$(\xi + \xi = 2 \,.\, \xi) = (\overset{\mathfrak{a}}{\smile}\, \xi = \mathfrak{a}),$$ [a]
$$(\xi + \xi = 2 \,.\, \xi) = (\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \xi),$$
$$(\xi + \xi = 2 \,.\, \xi) = (\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a});$$

but the first two conceptions contradict the fact that the reference of the '$\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$' occurring in

$$`\overset{\mathfrak{a}}{\smile}((\mathfrak{a} + \mathfrak{a} = 2 \,.\, \mathfrak{a}) = (\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}))'$$ [b]

is already established and must not be called into question again.

We now call that what follows a concavity with a *German letter*, which forms together with just this concavity the name of the truth-value for: that the value of the corresponding function for every argument is the True, the *scope* of the German letter standing over the concavity. Now, the *corresponding* function is determined by the rule:

1. All places, in which a German letter occurs in its own scope, but not within a subordinate scope of the same letter nor above a concavity, are related argument places, namely those of the corresponding function.

If, however, one wants to designate the truth-value of the function

$$(\xi + \xi = 2 \,.\, \xi) = (\overset{\mathfrak{a}}{\smile}\, \xi = \mathfrak{a})$$

having the True as value for every argument, then one will choose a different German letter:

$$\overset{\mathfrak{e}}{\smile}(\mathfrak{e} + \mathfrak{e} = 2 \,.\, \mathfrak{e}) = (\overset{\mathfrak{a}}{\smile}\, \mathfrak{e} = \mathfrak{a}).$$ [c]

| I capture this in the following rule: 14

[a]Typo in Frege in this and the following two formulae: he has a '+' instead of the rightmost occurrence of '='.

[b]Typo: Frege has a '+' instead of the rightmost occurrence of '='

[c]Thiel: typo in Frege: he misses out the outermost brackets that indicate the scope of '$\overset{\mathfrak{e}}{\smile}$'. However, according to Frege's definition of 'scope' the brackets are not required; hence, no typo after all. Compare also Frege's long footnote in §10.

2. If in the name of a function German letters already occur, within whose scopes lie argument places of this function, then a German letter distinct from these is to be chosen in order to form the corresponding expression of generality.

According to our specifications, one German letter is in general as good as any other, with the restriction, however, that the distinctness of these letters can be essential. For some German letters we will stipulate later a slightly different kind of use.

$$\text{‘}\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})\text{’}$$

refers to the same as

$$\text{‘}\overset{\mathfrak{a}}{\smile}(\!-\!\!-\, \Phi(\mathfrak{a}))\text{’}$$

and as

$$\text{‘}\!-\!\!-(\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a}))\text{’}$$

I therefore consider the horizontal stroke left and right of the concavity in ‘$\overset{\mathfrak{a}}{\smile}$’ as the *horizontal* in our special sense of the word, so that by the *fusing* of the horizontal we can immediately pass from ‘$-\!\!-(\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a}))$’ and ‘$\overset{\mathfrak{a}}{\smile}(-\!\!-\, \Phi(\mathfrak{a}))$’ to ‘$\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a})$’.

§9. If $\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$ is the True, we can, according to our previous specification (§3), also say that the function $\Phi(\xi)$ has the same value-range as the function $\Psi(\xi)$; that is: we can convert the generality of an equality into a value-range equality and *vice versa*. This possibility must be regarded as a logical law of which, incidentally, use has always been made, even if tacitly, whenever extensions of concepts were mentioned. The entire calculating logic of Leibniz and Boole rests upon it. One could perhaps regard this conversion as unimportant or even dispensable. Against this, I remind the reader, that in my *Grundlagen der Arithmetik*, I defined a cardinal number as the extension of a concept, and I had already then pointed out that the negative, irrational, in brief, all numbers are also to be defined as extensions of concepts. We can fix a simple sign for a value-range, and this is, e.g., how the name of the cardinal number Zero will be introduced. In contrast, in ‘$\overset{\mathfrak{a}}{\smile}\, \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$’ we cannot put a simple sign for ‘$\Phi(\mathfrak{a})$’, because the letter ‘$\mathfrak{a}$’ always has to occur in what can be put for ‘$\Phi(\mathfrak{a})$’, for example.

The transformation of the generality of an equality into a value-range equality must also be possible in our signs. Thus I write, e.g., for

$$\text{‘}\overset{\mathfrak{a}}{\smile}\, \mathfrak{a}^2 - \mathfrak{a} = \mathfrak{a}\,.\,(\mathfrak{a}-1)\text{’}$$
$$\text{‘}\grave{\varepsilon}(\varepsilon^2 - \varepsilon) = \grave{\alpha}(\alpha\,.\,(\alpha - 1))\text{’}$$

by understanding ‘$\grave{\varepsilon}(\varepsilon^2 - \varepsilon)$’ as the value-range of the function $\xi^2 - \xi$ and | 15

‘$\dot{\alpha}(\alpha\,.\,(\alpha{-}1))$’ as the value-range of the function $\xi\,.\,(\xi{-}1)$. Equally, $\dot{\varepsilon}(\varepsilon^2 = 4)$ is the value-range of the function $\xi^2 = 4$, or, as we can also say, the extension of the concept *square root of 4*.

If I say in general:

> let
>
> ‘$\dot{\varepsilon}\Phi(\varepsilon)$’
>
> refer to the value-range of the function $\Phi(\xi)$,

then this too requires supplementation, just like our explanation of ‘$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a})$’ above. Specifically, the question is which function is to be regarded as the *corresponding* function $\Phi(\xi)$ in each case. That $\dot{\varepsilon}(\varepsilon^2 - \varepsilon)$ is the value-range of the function $\xi^2 - \xi$ and not of $\xi^2 - \varepsilon$ nor of $\varepsilon^2 - \xi$ is readily understood because in our usage of the *small Greek vowel* neither ‘$\xi^2 - \varepsilon$’ nor ‘$\varepsilon^2 - \xi$’ would acquire a reference for any object whose name were inserted for ‘ξ’, or, as we can also put it, because those combinations of signs refer to no functions, but lack reference if detached from ‘$\dot{\varepsilon}$’. A combination of signs like ‘$\dot{\varepsilon}\Psi(\varepsilon, \dot{\varepsilon}\mathrm{X}(\varepsilon))$’ has to be judged similar to ‘$\overset{\mathfrak{a}}{\smile}\,\Psi(\mathfrak{a}, \overset{\mathfrak{a}}{\smile}\,\mathrm{X}(\mathfrak{a}))$’ in §8. The place under the smooth breathing is no more an *argument place* than the one above the concavity. Let us call the *scope* of a *small Greek vowel* that which follows this Greek letter with a smooth breathing, and together with it forms the name of the value-range of the *corresponding* function, so we can lay down the rule:

1. All places in which a small Greek vowel occurs in its own scope but not within a subordinate scope of the same letter nor with the smooth breathing, are related argument places, namely those of the corresponding function.

The latter is hereby determined. Accordingly, $\dot{\varepsilon}(\varepsilon = \dot{\varepsilon}(\varepsilon^2 - \varepsilon))$ is the value-range of the function $\xi = \dot{\varepsilon}(\varepsilon^2 - \varepsilon)$, and $\dot{\alpha}(\alpha = \dot{\varepsilon}(\varepsilon = \alpha))$ is the value-range of the function $\xi = \dot{\varepsilon}(\varepsilon = \xi)$. The following rule thus applies to the formation of a name for a value-range:

2. If small Greek vowels already occur in the name of a function, in whose scope argument places of this function lie, then one is to choose one that is different from those in order to form the name of the value-range of this function.

According to our specifications, one *small Greek vowel* is in general as good as any other, with the restriction, however, that the distinctness of these letters can be essential.

The introduction of the notation for value-ranges seems to me | one of 16
the most consequential additions to my concept-script that I made since my first publication on this subject matter. Thereby, also the domain of that which can occur as an argument of a function is extended. For example,

$\dot{\varepsilon}(\varepsilon^2 - \varepsilon) = \dot{\alpha}(\alpha\,.\,(\alpha - 1))$ is the value of the function $\xi = \dot{\alpha}(\alpha\,.\,(\alpha - 1))$ for the argument $\dot{\varepsilon}(\varepsilon^2 - \varepsilon)$.

§10. By presenting the combination of signs '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$' as co-referential with '$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', we have admittedly by no means yet completely fixed the reference of a name such as '$\dot{\varepsilon}\Phi(\varepsilon)$'. We have a way always to recognise a value-range as the same if it is designated by a name such as '$\dot{\varepsilon}\Phi(\varepsilon)$', whereby it is already recognisable as a value-range. However, we cannot decide yet whether an object that is not given to us as a value-range is a value-range or which function it may belong to; nor can we decide in general whether a given value-range has a given property if we do not know that this property is connected with a property of the corresponding function. If we assume that

$$\mathrm{X}(\xi)$$

is a function that never receives the same value for different arguments, then exactly the same criterion[a] for recognition holds for the objects whose names have the form '$\mathrm{X}(\dot{\varepsilon}\Phi(\varepsilon))$' as for the objects whose signs have the form '$\dot{\varepsilon}\Phi(\varepsilon)$'. For then '$\mathrm{X}(\dot{\varepsilon}\Phi(\varepsilon)) = \mathrm{X}(\dot{\alpha}\Psi(\alpha))$' too is co-referential with '$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'.[1] From this it follows that by equating the reference of '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$' with that of '$\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', the reference of a name such as '$\dot{\varepsilon}\Phi(\varepsilon)$' is by no means completely determined; at least if there is such a function $\mathrm{X}(\xi)$ whose value for a value-range as argument is not always equal to the value-range itself. Now, how is this indeterminacy resolved? By determining for every function, when introducing it, which value it receives for value-ranges as arguments, just as for all other arguments. Let us do this for the functions hitherto considered. These are the following:

$$\xi = \zeta,\ \text{—}\,\zeta,\ \top\,\zeta$$

The last one can be left out of consideration, since its argument may always be taken to be a truth value. It makes no difference whether one takes as argument an object or the value that the function $\text{—}\,\xi$ has for this object as argument. In addition, we can now reduce the function $\text{—}\,\xi$ to the function $\xi = \zeta$. For based on our stipulations the function $\xi = (\xi = \xi)$ has the same value as the function $\text{—}\,\xi$ for every argument; for the value of the function $\xi = \xi$ is the True for every argument. It follows from this that | the value of 17
the function $\xi = (\xi = \xi)$ is the True only for the True as argument, and that it is the False for all other arguments, just as for the function $\text{—}\,\xi$. After having thus reduced everything to the consideration of the function $\xi = \zeta$, we ask which values it has when a value-range appears as argument. Since so far we have only introduced the truth-values and value-ranges as objects,

[a] *Kennzeichen* — see introduction

[1] Thereby it is not said that the sense is the same.

the question can only be whether one of the truth-values might be a value-range. If that is not the case, then it is thereby also decided that the value of the function $\xi = \zeta$ is always the False when a truth-value is taken as one of its arguments and a value-range as the other. If, on the other hand, the True is at the same time the value-range of a function $\Phi(\xi)$, then it is thereby also decided what the value of the function $\xi = \zeta$ is in all cases where the True is taken as one of the arguments; and matters are similar if the False is at the same time the value-range of a certain function. Now, the question whether one of the truth-values is a value-range cannot possibly be decided on the basis of '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$' having the same reference as '$\overset{\mathfrak{a}}{\smile} \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'. It is possible to stipulate generally that '$\tilde{\eta}\Phi(\eta) = \tilde{\alpha}\Psi(\alpha)$' is to refer to the same as '$\overset{\mathfrak{a}}{\smile} \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', without it being possible to infer from that to the equality of $\dot{\varepsilon}\Phi(\varepsilon)$ and $\tilde{\eta}\Phi(\eta)$. We would then have, for example, a class of objects with names of the form '$\tilde{\eta}\Phi(\eta)$' for whose differentiation and recognition the same criterion would hold as for the value-ranges. We could now determine the function $\mathrm{X}(\xi)$ by saying that its value is to be the True for $\tilde{\eta}\Lambda(\eta)$ as argument, and it is to be $\tilde{\eta}\Lambda(\eta)$ for the True as argument; further, the value of the function, $\mathrm{X}(\xi)$, is to be the False for the argument $\tilde{\eta}\mathrm{M}(\eta)$, and it is to be $\tilde{\eta}\mathrm{M}(\eta)$ for the False as argument; for every other argument, the value of the function $\mathrm{X}(\xi)$[a] is to coincide with the argument itself. So, provided the functions $\Lambda(\xi)$ and $\mathrm{M}(\xi)$ do not always have the same value for the same argument, our function $\mathrm{X}(\xi)$ never has the same value for different arguments, and therefore '$\mathrm{X}(\tilde{\eta}\Phi(\eta)) = \mathrm{X}(\tilde{\alpha}\Psi(\alpha))$' is then also always co-referential with '$\overset{\mathfrak{a}}{\smile} \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'. The objects whose names would be of the form '$\mathrm{X}(\tilde{\eta}\Phi(\eta))$' would then also be recognised by the same means as the value-ranges, and $\mathrm{X}(\tilde{\eta}\Lambda(\eta))$ would be the True and $\mathrm{X}(\tilde{\eta}\mathrm{M}(\eta))$ would be the False. Thus, without contradicting our equating '$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\varepsilon}\Psi(\varepsilon)$' with '$\overset{\mathfrak{a}}{\smile} \Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', it is always possible to determine that an arbitrary value-range be the True and another arbitrary value-range be the False. Let us therefore stipulate that $\dot{\varepsilon}(—\, \varepsilon)$ be the True and that $\dot{\varepsilon}(\varepsilon = (\top\!\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}))$ be the False. $\dot{\varepsilon}(—\, \varepsilon)$ is the value-range of the function $—\, \xi$, whose value is the True only if the argument is the True, and whose value is the False for all other arguments. All functions of which this holds have | the same value- 18
range and, according to our stipulation, this is the True. Thus $—\, \dot{\varepsilon}\Phi(\varepsilon)$ is the True only if the function $\Phi(\xi)$ is a concept under which only the True falls; in all other cases $—\, \dot{\varepsilon}\Phi(\varepsilon)$ is the False. Further, $\dot{\varepsilon}(\varepsilon = (\top\!\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}))$ is the value-range of the function, $\xi = (\top\!\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a})$, whose value is the True only if the argument is the False, and whose value is the False for all other arguments. All functions of which this holds have the same value-range and, according to our stipulation, this is the False. Every concept, therefore, under which the False and only it falls, has as its extension the

[a]Typo in Frege as noted by Thiel: Frege has '$\Phi(\xi)$' instead of '$\mathrm{X}(\xi)$'.

False.[1]

We have hereby determined the *value-ranges* as far as is possible here. Only when the further issue arises of introducing a function that is not completely reducible to the functions already known will we be able to stipulate what values it should have for value-ranges as arguments; and this can then be viewed as a determination of the value-ranges as well as of that function.

§11. Indeed we do still require such functions. If the equating of '$\dot{\varepsilon}(\Delta = \varepsilon)$' with '$\Delta$' could be maintained generally,[2] then we would have a substitute
for the | definite article in language in the form '$\dot{\varepsilon}\Phi(\varepsilon)$'. For, if we assumed 19
that $\Phi(\xi)$ were a concept under which the object Δ and only this fell, then $\overset{\mathfrak{a}}{\smile}\,\Phi(\mathfrak{a}) = (\Delta = \mathfrak{a})$ would be the True and hence also $\dot{\varepsilon}\Phi(\varepsilon) = \dot{\varepsilon}(\Delta = \varepsilon)$ would be the True, and following our equating of '$\dot{\varepsilon}(\Delta = \varepsilon)$' and '$\Delta$', $\dot{\varepsilon}\Phi(\varepsilon)$ would be the same as Δ; i.e., in case $\Phi(\xi)$ were a concept under which one and only one object fell, '$\dot{\varepsilon}\Phi(\varepsilon)$' would designate this object. This is admittedly not possible, because the former equation had to be abandoned in its full generality; nevertheless we can help ourselves by introducing the

[1] It suggests itself to generalise our stipulation so that every object is conceived as a value-range, namely, as the extension of a concept under which it falls as the only object. A concept under which only the object Δ falls is $\Delta = \xi$. We attempt the stipulation: let $\dot{\varepsilon}(\Delta = \varepsilon)$ be the same as Δ. Such a stipulation is possible for every object that is given to us independently of value-ranges, for the same reason that we have seen for truth-values. But before we may generalise this stipulation, the question arises whether it is not in contradiction with our criterion for recognising value-ranges if we take an object for Δ which is already given to us as a value-range. It is out of the question to allow it to hold only for such objects which are not given to us as value-ranges, because the way an object is given must not be regarded as its immutable property, since the same object can be given in different ways. Thus, if we insert '$\dot{\alpha}\Phi(\alpha)$' for 'Δ' we obtain

$$`\dot{\varepsilon}(\dot{\alpha}\Phi(\alpha) = \varepsilon) = \dot{\alpha}\Phi(\alpha)'$$

and this would be co-referential with

$$`\overset{\mathfrak{a}}{\smile}\,(\dot{\alpha}\Phi(\alpha) = \mathfrak{a}) = \Phi(\mathfrak{a})',$$

which, however, only refers to the True, if $\Phi(\xi)$ is a concept under which only a single object falls, namely $\dot{\alpha}\Phi(\alpha)$. Since this is not necessary, our stipulation cannot be upheld in its generality.

The equation '$\dot{\varepsilon}(\Delta = \varepsilon) = \Delta$' with which we attempted this stipulation, is a special case of '$\dot{\varepsilon}\Omega(\varepsilon, \Delta) = \Delta$', and one can ask how the function $\Omega(\xi, \zeta)$ would have to be constituted, so that it could generally be specified that Δ be the same as $\dot{\varepsilon}\Omega(\varepsilon, \Delta)$. Then

$$\dot{\varepsilon}\Omega(\varepsilon, \dot{\alpha}\Phi(\alpha)) = \dot{\alpha}\Phi(\alpha)$$

also has to be the True, and thus also

$$\overset{\mathfrak{a}}{\smile}\,\Omega(\mathfrak{a}, \dot{\alpha}\Phi(\alpha)) = \Phi(\mathfrak{a}),$$

no matter what function $\Phi(\xi)$ might be. We shall later be acquainted with a function having this property in $\xi \frown \zeta$; however we shall define it with the aid of the value-range, so that it cannot be of use for us here.

[2] Compare note 1.

function

$$\backslash\xi$$

with the specification to distinguish two cases:

1) if, for the argument, there is an object Δ such that $\dot{\varepsilon}(\Delta = \varepsilon)$ is the argument, then the value of the function $\backslash\xi$ is to be Δ itself;

2) if, for the argument, there is no object Δ such that $\dot{\varepsilon}(\Delta = \varepsilon)$ is the argument, then the argument itself is to be the value of the function $\backslash\xi$.

Accordingly, $\backslash\dot{\varepsilon}(\Delta = \varepsilon) = \Delta$ is the True, and then '$\backslash\dot{\varepsilon}\Phi(\varepsilon)$' refers to the object which falls under the concept $\Phi(\xi)$, if $\Phi(\xi)$ is a concept under which one and only one object falls; in all other cases '$\backslash\dot{\varepsilon}\Phi(\varepsilon)$' refers to the same as '$\dot{\varepsilon}\Phi(\varepsilon)$'. So, e.g., $2 = \backslash\dot{\varepsilon}(\varepsilon + 3 = 5)$ is the True, because 2 is the only object that falls under the concept,

that which increased by 3 yields 5

presupposing here a suitable definition of the plus-sign. $\dot{\varepsilon}(\varepsilon^2 = 1) = \backslash\dot{\varepsilon}(\varepsilon^2 = 1)$ is the True, because not just one object falls under the concept, *square-root of 1.* $\dot{\varepsilon}(\top\, \varepsilon = \varepsilon) = \backslash\dot{\varepsilon}(\top\, \varepsilon = \varepsilon)$ is the True because no object falls under the concept *not equal to itself.* $\dot{\varepsilon}(\varepsilon + 3) = \backslash\dot{\varepsilon}(\varepsilon + 3)$ is the True because the function $\xi + 3$ is not a concept.

Here, then, we have a substitute for the definite article of language, which serves to form proper names out of concept-words. For example, out of the words

'positive square-root of 2',

that refer to a concept, we form the proper name

'the positive square-root of 2'.

Here is a logical risk. For if we were to form out of the words 'square-root of 2' the proper name 'the square-root of 2', we would commit a logical error, since this proper name would be, without further stipulation, ambiguous[1] and just for that reason without reference. If there were no irrational numbers, as has indeed been asserted, then the proper name 'the positive square-root of 2' would also be without reference, at least | according to the 20
immediate sense of the word, without special stipulation. And if we were specifically to assign a reference to this proper name, then this would have no connection with its formation and it would not be permissible to infer that it was a positive square-root of 2, and yet we would be all too inclined to conclude that. This risk carried by the definite article is now avoided

[1] I am here taking for granted that there are negative and irrational numbers.

altogether, since '$\backslash\grave{\varepsilon}\Phi(\varepsilon)$' always has a reference, whether the function $\Phi(\xi)$ is not a concept, or a concept under which more than one or no object falls, or whether it is a concept under which one and only one object falls.

§12. Next, in order to be able to designate the subordination of concepts and other important relations, I introduce the function with two arguments

$$\begin{array}{l} \top\; \xi \\ \llcorner\; \zeta \end{array}$$

by means of the specification that its value shall be the False if the True is taken as the ζ-argument, while any object that is not the True is taken as ξ-argument; that in all other cases the value of the function shall be the True. According to this and the previous stipulations, the value of this function is also determined for value-ranges as arguments. It follows that

$$\begin{array}{l} \top\; \Gamma \\ \llcorner\; \Delta \end{array}$$

is the same as

$$-\left(\begin{array}{l} \top\; (-\,\Gamma) \\ \llcorner\; (-\,\Delta) \end{array}\right)$$

and therefore that in

$$\text{`}\begin{array}{l} \top\; \Gamma \\ \llcorner\; \Delta\text{'} \end{array}$$

we can regard the horizontal stroke before 'Δ', as well as each of the two parts of the upper horizontal stroke partitioned by the vertical, as the *horizontal* in our particular sense. We speak here, just as previously, of the *fusion of horizontals*. I call the vertical stroke the *conditional-stroke*. It may be lengthened as required.

The following propositions hold:

$$\text{`}\begin{array}{l} \vdash\!\top\; 3^2 > 2 \\ \quad\llcorner\; 3 \;\; > 2\text{'} \end{array}\;;\quad \text{`}\begin{array}{l} \vdash\!\top\; 2^2 > 2 \\ \quad\llcorner\; 2 \;\; > 2\text{'} \end{array}\;;\quad \text{`}\begin{array}{l} \vdash\!\top\; 1^2 > 2 \\ \quad\llcorner\; 1 \;\; > 2\text{'} \end{array}\,.$$

The function $\begin{array}{l} \top\!\top\; \xi \\ \;\llcorner\; \zeta \end{array}$, or $\begin{array}{l} \mp\!\top\; \xi \\ \;\;\llcorner\; \zeta \end{array}$, always has the True as value, when the function $\begin{array}{l} \top\; \xi \\ \llcorner\; \zeta \end{array}$ has the False,[a] and conversely. Hence $\begin{array}{l} \top\!\top\; \Gamma \\ \;\llcorner\; \Delta \end{array}$ is | the True if and only if Δ 21
is the True and Γ is not the True. Accordingly,

$$\begin{array}{l} \vdash\!\top\!\top\; 2 > 3 \\ \quad\;\llcorner\; 2 + 3 = 5 \end{array}$$

[a]Frege: *das Wahre* — error noted by Frege in vol. 2 in the *corrigenda* for vol. 1.

in words: 2 is not greater than 3 *and* the sum of 2 and 3 is 5.

$$\vdash\!\!\top\!\!\top\!\!\top 3 > 2 \quad \llcorner\, 2 + 3 = 5$$

in words: 3 is greater than 2 *and* the sum of 2 and 3 is 5. For $\top\!\!\top\, 3 > 2 \;\; \llcorner\, 2+3=5$ is the value of the function, $\top\!\!\top\, \xi \;\; \llcorner\, \zeta$, when $\top\, 3 > 2$ is the ξ-argument, and 2+3=5 is the ζ-argument.

$$\vdash\!\!\top\!\!\top\, 2^3 = 3^2 \quad \llcorner\!\!\top\, 1^2 = 2^1$$

in words: *neither* is the third power of 2 the second power of 3, *nor* is the second power of 1 the first power of 2.

By way of the propositions

$$`\vdash\!\!\top\, 3^2 > 3 \;\; \llcorner\!\!\top\, 3 < 3\text{'} \;;\quad `\vdash\!\!\top\, 2^2 > 3 \;\; \llcorner\!\!\top\, 2 < 3\text{'} \;;\quad `\vdash\!\!\top\, 1^2 > 3 \;\; \llcorner\!\!\top\, 1 < 3\text{'}$$

one has the following

$$`\vdash\!\!\top\!\!\top\!\!\top\, 3^2 > 3 \;\; \llcorner\!\!\top\, 3 < 3\text{'} \;;\quad `\vdash\!\!\top\!\!\top\!\!\top\, 2^2 > 3 \;\; \llcorner\!\!\top\, 2 < 3\text{'} \;;\quad `\vdash\!\!\top\!\!\top\!\!\top\, 1^2 > 3 \;\; \llcorner\!\!\top\, 1 < 3\text{'} .$$

Now, since $\top\!\!\top\, 1^2 > 3 \;\; \llcorner\!\!\top\, 1 < 3$ is the truth-value of: that neither is the square of 1 greater than 3, nor is 1 less than 3, this is negated by our last proposition, so it asserts at least one of the two is true, that the square of 1 is greater than 3 *or* that 1 is less than 3. One can see from these examples how the '*and*' of language, when it connects propositions, the '*neither — nor*', and the '*or*' between propositions, are to be rendered.

One can insert into '$\top\, \xi \;\; \llcorner\, \Delta$' any proper name for '$\xi$', even for example '$\top\, \Theta \;\; \llcorner\, \Lambda$'. Thus we obtain

$$`\top\left(\top\,\Theta \;\; \llcorner\,\Lambda\right) \;\; \llcorner\,\Delta\text{'},$$

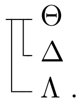
wherein we can now *fuse* the horizontals:

$$`\top\, \Theta \;\; \llcorner\, \Lambda \;\; \llcorner\, \Delta\text{'}$$

| This refers to the False if Δ is the True and $\top\, \Theta \;\; \llcorner\, \Lambda$ is not the True; i.e., in 22
this case, if $\top\, \Theta \;\; \llcorner\, \Lambda$ is the False. The latter, however, is the case if and only if Λ is the True and Θ is not the True. Thus,

is the False if Δ and Λ are the True while Θ is not the True; in all other cases it is the True. From this follows the permutability of Λ and Δ:

is the same truth-value as



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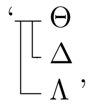
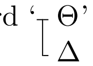
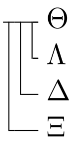
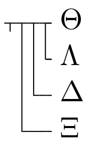


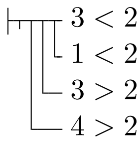
we shall term ‘— Θ’ *supercomponent*, ‘— Δ’ and ‘— Λ’ *subcomponents*. We can, however, also regard  as *supercomponent* and ‘— Λ’ alone as *subcomponent*. The subcomponents are therefore *permutable*. Likewise, we can see that



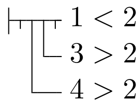
is the False if and only if Λ, Δ, and Ξ are all the True, while Θ is not the True. In all other cases it is the True. Once more we thus have the *permutability of the subcomponents*, ‘— Λ’, ‘— Δ’, ‘— Ξ’. Strictly speaking, this permutability must be proven for each occurring case, and I have done this for some cases in my little book *Begriffsschrift*, in such a way that it will be straightforward to treat all cases accordingly. In order not to be tied up in excessive prolixity, I will here assume this permutability to be generally granted and in what follows make use of it without further reminder. | 23



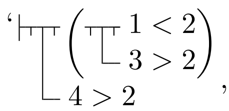
is the True, if and only if Λ, Δ, and Ξ are all the True, while Θ is not the True. Thus,



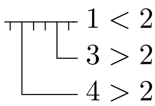
in words: 3 is not less than 2 and 1 is less than 2 and 3 is greater than 2 and 4 is greater than 2;



in words: 1 is less than 2 and 3 is greater than 2 and 4 is greater than 2. One can think of the latter as dissected thus:[a]

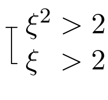


The negation strokes between the conditional-strokes cancel each other and the horizontals may be fused. We have in

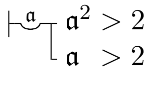


the value of the function $\begin{smallmatrix}\xi\\ \zeta\end{smallmatrix}$ for $\begin{smallmatrix}1 < 2\\ 3 > 2\end{smallmatrix}$ as ξ-argument and $4 > 2$ as the ζ-argument, wherein $\begin{smallmatrix}1 < 2\\ 3 > 2\end{smallmatrix}$ is, in turn, the value of the same function for $1 < 2$ as ξ-argument and $3 > 2$ as ζ-argument.

§13. To justify the name, 'conditional-stroke', I point out that the names '$\begin{smallmatrix}3^2 > 2\\ 3 > 2\end{smallmatrix}$', '$\begin{smallmatrix}2^2 > 2\\ 2 > 2\end{smallmatrix}$', '$\begin{smallmatrix}1^2 > 2\\ 1 > 2\end{smallmatrix}$' result from '$\begin{smallmatrix}\xi^2 > 2\\ \xi > 2\end{smallmatrix}$' by replacing '$\xi$' with '3', '2' and '1'. If we now use the sign, '>', in such a way that '$\Gamma > \Delta$' refers to the True if Γ and Δ are real numbers and Γ is greater than Δ, and that in all other cases '$\Gamma > \Delta$' refers to the False; if we assume further that the notation 'Γ^2' is explained so that it has a reference whenever Γ is an object, then the value of the function

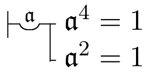


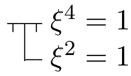
| is the True for every argument; hence 24

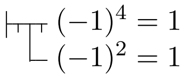


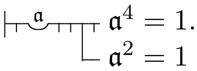
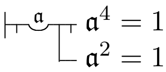
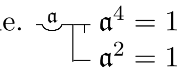
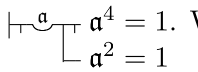
[a]Typo in Frege, noted by Thiel: instead of $3 > 2$ he has $4 > 2$ in the formula below.

in words: *if* something is greater than 2 then its square is also greater than 2. So also



in words: *if* the square of something is 1 then its fourth power is also 1. One can, however, also say: *every* square root of 1 is also a fourth root of 1; or: *all* square roots of 1 are also fourth roots of 1.[1] Here we have the *subordination* of a concept under a concept, a *universal* affirmative proposition. We have called any function of one argument whose value is always a truth-value a concept. Such functions are here $\xi^4 = 1$ and $\xi^2 = 1$; the latter is the *subordinate*, the former the *superordinate* concept.  is composed from these concepts as characteristic marks. Under this concept falls, e.g., the number -1:[a]



in words: -1 is square root of 1 and fourth root of 1. We have seen in §8 how the 'there is' of ordinary language is rendered. We apply this to say that there is something that is square root of 1 and fourth root of 1:  Obviously the two negation-strokes cancel each other out here:  Let us have a look at this from another side.  is the truth-value of: that, if anything is a square root of 1, then it is not a fourth root of 1; or, as we can also say, *no* square root of 1 is a fourth root of 1. This truth-value is the False, and hence:  We have here the negation of a *universal* negative proposition, i.e., a *particular* affirmative proposition,[2] for which we can also say: '*some* square roots of 1 | are fourth 25

[1]One easily connects this with the accompanying thought that there is something that is a square root of 1. This must be kept entirely at a distance. Likewise, the accompanying thought that there is more than one square root of 1 is likewise to be fended off here.

[a]Typo in Frege, noted by Thiel: Frege has the content-stroke '—' instead of the minus-sign '−' here and in the lines below.

[2]The *particular* affirmative proposition, on the one hand, says less than the *universal* affirmative one, but, on the other hand, as is easily overlooked, also say more, since it asserts the instantiation of the concept, while subordination also, and indeed always, occurs with empty concepts. Some logicians seem to assume concepts to be instantiated without further ado and to overlook completely the very important case of the empty concept, perhaps because they, quite wrongly, do not acknowledge empty concepts as legitimate. | It is because of this that I do not use the expressions 'subordination', 'universal affirmative', 'particular affirmative' in exactly the same sense as these logicians, and so arrive at pronouncements which they will be wrongly inclined to regard as false.

roots of 1', where, however, the form of the plural is not to be understood as requiring that there must be more than one.

$$\vdash\!\overset{\mathfrak{a}}{\smile}\!\!\top\!\!\!\top\begin{array}{l}\mathfrak{a}^4 = 1\\ \llcorner\, \mathfrak{a}^3 = 1\end{array}$$

in words: there is at least one cube root of 1 which is also a fourth root of 1; or: *some* — or at least one — cube root of 1 is a fourth root of 1.

In our symbolism, the proposition-connecting 'and' appears less simple than the function-name '$\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}$', for which a simple ordinary language expression is wanting. The relationship present in ordinary language easily seems more natural and appropriate, because it is familiar. However, which is simpler from a logical point of view is not easy to say: using 'and' and negation one can explain our '$\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}$', but also conversely using the function name '$\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}$' and the negation-stroke one can explain 'and'. Obviously, for example, '$\begin{array}{l}\vdash\!\top\, 2+3=5\\ \quad\llcorner\, 2+2=4\end{array}$' says less than '$\begin{array}{l}\vdash\!\top\!\!\top\, 2+3=5\\ \qquad\llcorner\, 2+2=4\end{array}$'[a] and could therefore be considered simpler. The ultimate reason for the introduction of '$\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}$' is the ease and perspicuity with which one can thereby represent inference, to which we now proceed.

Inferences and Consequences

§14. From the propositions '$\begin{array}{l}\vdash\!\top\,\Gamma\\ \quad\llcorner\,\Delta\end{array}$' and '$\vdash \Delta$' one can infer: '$\vdash \Gamma$'; for if Γ were not the True, then, since Δ is the True, $\begin{array}{l}\top\,\Gamma\\ \llcorner\,\Delta\end{array}$ would be the False. To each proposition put forward in concept-script symbolism, if it is to be used later in a further conduct of proof, I will assign a *label* for the purpose of citation. Accordingly, if the proposition '$\begin{array}{l}\vdash\!\top\,\Gamma\\ \quad\llcorner\,\Delta\end{array}$' has received the label 'α' and '$\vdash \Delta$' the label 'β', then I will write the inference either as

$$\text{`}\quad\begin{array}{l}\vdash\!\top\,\Gamma\\ \quad\llcorner\,\Delta\end{array} \qquad\qquad \text{or as} \qquad\qquad \text{`}\quad \vdash \Delta$$

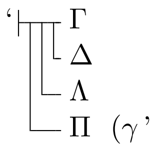
$$(\beta)\!::\!\frac{\qquad}{\vdash \Gamma}\ \text{'} \qquad\qquad\qquad (\alpha)\!:\!\frac{\vdash \Delta}{\vdash \Gamma}\ \text{'}$$

with double colon   with single colon.

[a] Frege has '$2 + 3 = 4$' in the subcomponent — Thiel notes the typo and corrects into '$2 + 2 = 4$'.

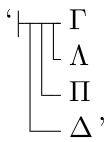
| This is the sole mode of inference that I used in my *Begriffsschrift*,[a] and one can even manage with it alone. The demand of scientific parsimony would now usually require this; but considerations of practicality pull in the opposite direction, and here, where I will have to form long chains of inferences, I will have to make some concessions. For an inordinate lengthiness would result if I were not to allow some other modes of inference, as already anticipated in the preface of my little work. 26a

If we are given the propositions

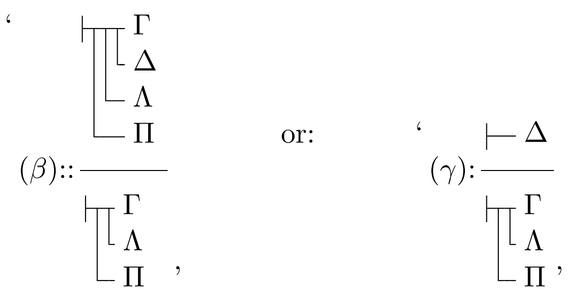


and ‘⊦ Δ (β’

then we cannot immediately make the inference as above but only after having transformed (γ) by making use of the permutability of the subcomponents thus:



However, in order to avoid excessive elaboration, I will not write this out explicitly but rather write immediately



| in which the subcomponents of the conclusion could also be ordered differently. 26b

> *If a subcomponent of a proposition differs from a second proposition only in lacking the judgement-stroke, then one may infer a proposition which results from the first by suppressing that subcomponent.*

We also combine two inferences of this kind as can be seen from what follows. Let another proposition be given, ‘⊦ Λ (ϱ’. Then we write the double inference in this way:

[a] *Translators' Note*: Not emphasised in the original. Frege seems to be referring to the book (see end of passage), rather than the system, though, so it should be.

$$
\text{‘}\quad
\begin{array}{l}
\vdash\!\!\top\!\!\top\!\!\top\; \Gamma \\
\quad\;\;\llcorner\; \Delta \\
\quad\;\llcorner\; \Lambda \\
\quad\llcorner\; \Pi \\
(\beta,\varrho)::\!=\!=\!=\!=\!= \\
\vdash\!\!\top\; \Gamma \\
\;\;\llcorner\; \Pi
\end{array}
\quad\text{’}
$$

§15. The following mode of inference is a little bit less simple. From the two propositions

$$
\text{‘}\begin{array}{l}\vdash\!\!\top\;\Gamma\\ \;\;\llcorner\;\Delta\end{array}\;(\alpha\text{’}
\qquad\text{and}\qquad
\text{‘}\begin{array}{l}\vdash\!\!\top\;\Delta\\ \;\;\llcorner\;\Theta\end{array}\;(\delta\text{’}
$$

we can infer the proposition ‘$\begin{array}{l}\vdash\!\!\top\;\Gamma\\ \;\;\llcorner\;\Theta\end{array}$’. For $\begin{array}{l}\top\;\Gamma\\ \llcorner\;\Theta\end{array}$[b] is only the False if Θ is the True and Γ is not the True. However if Θ is the True then also Δ must be the True because otherwise $\begin{array}{l}\top\;\Delta\\ \llcorner\;\Theta\end{array}$ would be the False. If, however, Δ is the True, then were Γ not the True, | then $\begin{array}{l}\top\;\Gamma\\ \llcorner\;\Delta\end{array}$ would be the False. The case in 27a
which $\begin{array}{l}\top\;\Gamma\\ \llcorner\;\Theta\end{array}$ is the False cannot, therefore, occur, and $\begin{array}{l}\top\;\Gamma\\ \llcorner\;\Theta\end{array}$ is the True.

This inference I write either like this:

$$
\text{‘}\quad
\begin{array}{l}
\vdash\!\!\top\;\Gamma\\
\;\;\llcorner\;\Delta\\
(\delta)::-\,-\,-\\
\vdash\!\!\top\;\Gamma\\
\;\;\llcorner\;\Theta
\end{array}\;\text{’}
\qquad\text{or like this:}\qquad
\text{‘}\quad
\begin{array}{l}
\vdash\!\!\top\;\Delta\\
\;\;\llcorner\;\Theta\\
(\alpha):-\,-\,-\\
\vdash\!\!\top\;\Gamma\\
\;\;\llcorner\;\Theta
\end{array}\;\text{’}
$$

If instead of the proposition (α) we have as premise the proposition labeled ‘γ’ in §14, then we should have to carry out a transformation before making the inference, as we did there. Yet, for the sake of brevity, we perform this, as above, mentally and write:

$$
\text{‘}\quad
\begin{array}{l}
\vdash\!\!\top\!\!\top\!\!\top\;\Gamma\\
\quad\;\;\llcorner\;\Delta\\
\quad\;\llcorner\;\Lambda\\
\quad\llcorner\;\Pi\\
(\delta)::-\,-\,-\,-\\
\vdash\!\!\top\!\!\top\!\!\top\;\Gamma\\
\quad\;\;\llcorner\;\Theta\\
\quad\;\llcorner\;\Lambda\\
\quad\llcorner\;\Pi
\end{array}\;\text{’}
\qquad\text{or:}\qquad
\text{‘}\quad
\begin{array}{l}
\vdash\!\!\top\;\Delta\\
\;\;\llcorner\;\Theta\\
(\gamma):-\,-\,-\,-\\
\vdash\!\!\top\!\!\top\!\!\top\;\Gamma\\
\quad\;\;\llcorner\;\Theta\\
\quad\;\llcorner\;\Lambda\\
\quad\llcorner\;\Pi
\end{array}\;\text{’}
$$

[b] Typo, noted by Thiel: Frege uses ‘$\vdash$’ in front of the formula; this is to be deleted.

$\neg\Gamma \to \neg\Delta$ is the False if $\neg\Gamma$ is the True and $\neg\Delta$ is not the True; i.e. if — Γ is the False and Δ is the True. In all other cases $\neg\Gamma \to \neg\Delta$ is the True. The same holds, however, also for $\Delta \to \Gamma$, so that the functions $\neg\xi \to \neg\zeta$ and $\zeta \to \xi$ | always 27b
have the same value for the same arguments. Equally, the functions $\neg\xi \to \zeta$ and $\neg\zeta \to \xi$ always have the same value for the same arguments. One may reduce this case to the previous one by putting '$\neg\zeta$' for 'ζ' and canceling adjacent negation-strokes. Further, the functions $\zeta \to \neg\xi$ and $\xi \to \neg\zeta$ always have the same value for the same arguments. We can therefore pass from the proposition '$\Delta \to \Gamma$' to the proposition '$\neg\Gamma \to \neg\Delta$' and conversely from the latter to the former. We write these transitions as follows:

$$\text{`}\vdash \Delta \to \Gamma \;\times\; \vdash \neg\Gamma \to \neg\Delta\text{'} \quad \text{and} \quad \text{`}\vdash \neg\Gamma \to \neg\Delta \;\times\; \vdash \Delta \to \Gamma\text{'}.$$

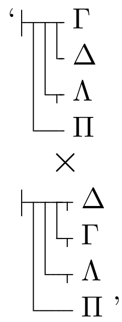
Likewise also:

$$\text{`}\vdash \neg\Delta \to \Gamma \;\times\; \vdash \neg\Gamma \to \Delta\text{'} \quad \text{and} \quad \text{`}\vdash \Delta \to \neg\Gamma \;\times\; \vdash \Gamma \to \neg\Delta\text{'},$$

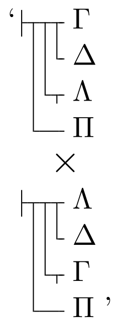
cases which are reducible to the first of the above by the elimination of negation-strokes. We can capture this in a rule thus:

> One may permute a subcomponent with a supercomponent provided one simultaneously *reverses* the truth-value of each.

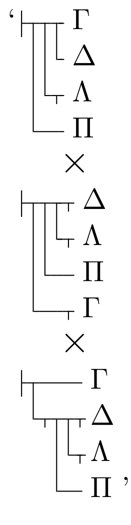
We call this transition *contraposition*. However, there may also be several | 28a
subcomponents present. In that case we have the transition



However, by tacit appeal to the permutability of the subcomponents, we may also write:

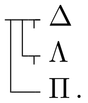


By two steps of contraposition all subcomponents may be combined in one, as follows:

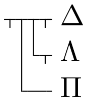


| For in the second contraposition we regard 28b

as supercomponent and ‘⊤ Γ’ as subcomponent. Let ‘Θ’ be an abbreviation for the truth-value

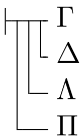
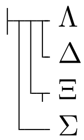


The penultimate proposition then becomes ‘⊢ Θ, ⊤ Γ’ from which ‘⊢ Γ, ⊤ Θ’ follows. If we then again insert for ‘Θ’ the unabbreviated expression, we obtain the conclusion. As can be seen from §12, we have in

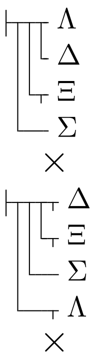


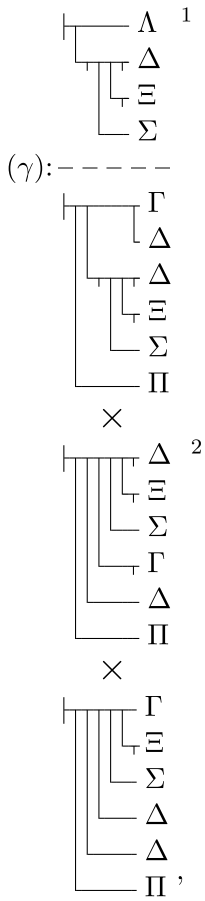
the truth-value of: that Δ is the True, Λ is not the True and Π is the True.

If we take the propositions

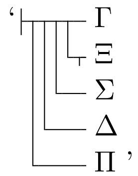
 (γ’ and  (ε’

as given, we may make the inference as follows: we first combine the subcomponents of (ε):

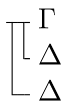
‘  29a



This can be simplified by writing 'Δ' only once:



for



is always the same truth-value as 

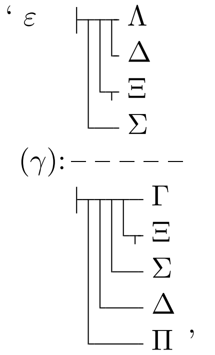
| A subcomponent occurring twice need only be written once. 29b

[1]Here we can infer in the same way as at the start of this paragraph since this proposition has the same form as (δ) there.

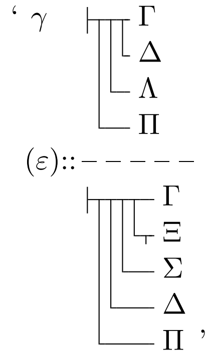
[2]We now resolve the complex subcomponent again.

We call this the *fusion* of equal subcomponents.

I will write this transition as:



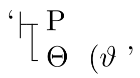
or like this:



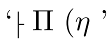
and put forth the following rule:

> *If the same combination of signs occurs in one proposition as supercomponent and in another as subcomponent, then a proposition may be inferred in which the supercomponent of the latter features as supercomponent and all subcomponents of both, save that mentioned, feature as subcomponents. However, subcomponents that occur in both need only be written once.*

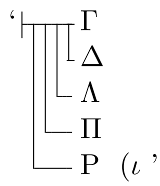
In a manner similar to that of §14, we can here combine two inferences. For example, if in addition to (ε) we are given the propositions



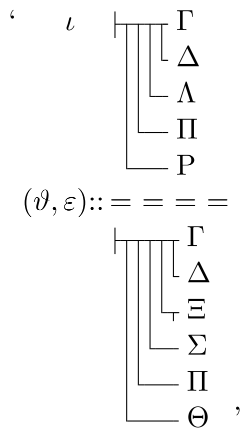
and



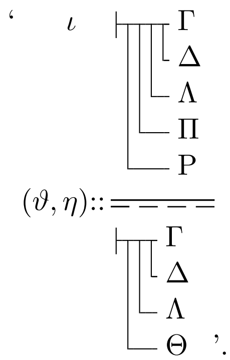
and



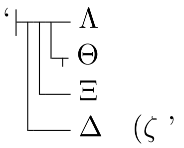
then we can write 30a



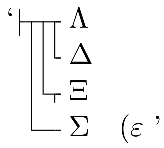
and



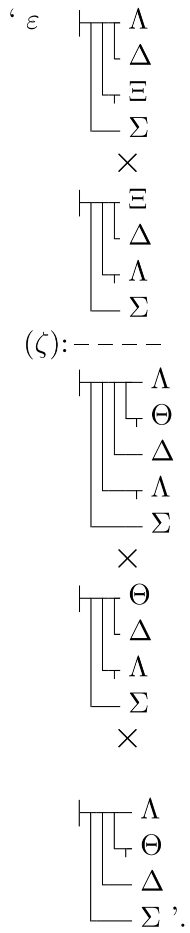
§16. Assume the propositions



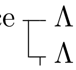
and

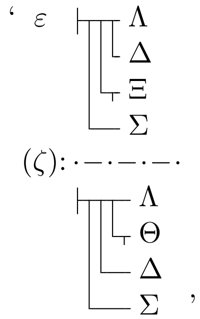


are given, we can reduce this case to the one just treated as follows:



30b

The purpose of these two contrapositions is to do away with one occurrence of ‘⊤ Λ’ by fusion of equal subcomponents. That  is always the same truth-value as — Λ can also be seen immediately since  is the False when ⊤ Λ is the True and Λ is not the True, and is otherwise the True, for the latter condition contains the first. However, — Λ is also the False when Λ is not the True, and is otherwise the True. We now abbreviate this transition as:



and articulate the rule like this:

> *If two propositions agree in their supercomponents while a subcomponent of the one differs from a subcomponent of the other only by a prefixed negation stroke, then we can infer a proposition in which the com|mon supercomponent features as super-* 31a
> *component, and all subcomponents of both propositions with the exception of the two mentioned feature as subcomponents. In this, subcomponents which occur in both propositions need only be written down once* (fusion of equal subcomponents).

§17. Let us now see how the inference called 'Barbara' in logic fits in here. From the two propositions:

'All square roots of 1 are fourth roots of 1'

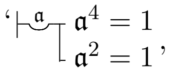
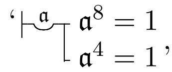
and

'All fourth roots of 1 are eighth roots of 1'

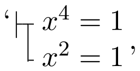
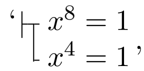
we can infer:

'All square roots of 1 are eighth roots of 1'

If we now write the premises thus:

 and 

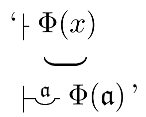
then we cannot apply our modes of inference; however we can if we write the premises as follows:

 and 

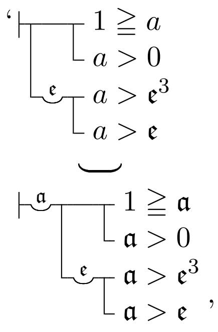
Here we have the case of §15. Above we attempted to express generality in this way using a *Roman letter*, but abandoned it because we observed that the scope of generality would not be adequately demarcated. We now address this concern by stipulating that the *scope* of | a *Roman letter* should 31b include everything that occurs in the proposition apart from the judgement-stroke.[1] Accordingly, one can never express the negation of a generality by means of a Roman letter, although we can express the generality of a negation. An ambiguity is thus no longer present. Nevertheless, it is clear that the expression of generality with German letters and concavity is not rendered superfluous. Our stipulation regarding the *scope* of a *Roman letter* is only to demarcate its narrowest extent and not its widest. It thus remains permissible to let the scope extend to multiple propositions so that the Roman letters are suitable to serve in inferences in which the German letters, with their strict demarcation of scope, cannot serve. So, when our premisses are ‘$\vdash\!\top\, x^8 = 1 \;\llcorner\, x^4 = 1$’ and ‘$\vdash\!\top\, x^4 = 1 \;\llcorner\, x^2 = 1$’, then in order to make the inference to the conclusion ‘$\vdash\!\top\, x^8 = 1 \;\llcorner\, x^2 = 1$’,[a] we temporarily expand the scope of ‘x’ to include both premisses and conclusion, although each of these propositions holds even without this extension.

We do not say of a Roman letter that it *refers to* an object but that it *indicates* an object.[b] | Likewise, we say that a German letter *indicates* an object where it does not stand over a concavity. 32a

A proposition with a Roman letter can always be transformed into one with a German letter whose concavity is separated from the judgement-stroke only by a horizontal. We write such transition thus:



In doing so, the second rule of §8 must be observed, as in the following example where ‘$\mathfrak{e}$’ may not be chosen as the German letter newly to be introduced.



[1]The use of Roman letters is hereby explained only for the case in which a judgement-stroke occurs. This, however, is always the case in a pure concept-development; since then we always progress from proposition to proposition.

[a]Typos in Frege — first premise and conclusion have to be swapped — we corrected. [Thiel suggests a different way of resolution, our way preserves the standard form of the Barbara syllogism.]

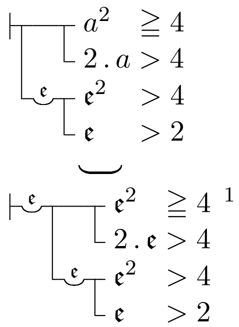
[b]*Translators' Note*: refer back to *an-/bedeuten* footnote above.

In the transition from a Roman to a German letter the following case also must be mention. Consider the proposition '$\vdash\!\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$'; where '$\Gamma$' is a proper name and '$\Phi(\xi)$' a function-name. $\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$ is the False if the function $\begin{array}{l}\top\,\Phi(\xi)\\ \llcorner\,\Gamma\end{array}$ has the False as value for any particular argument. This case obtains if Γ is the True and the value | of the function $-\!\!-\,\Phi(\xi)$ is the False for some 32b
argument. In all other cases $\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$ is the True. '$\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$' thus says that either Γ is not the True or that the value of the function $\Phi(\xi)$ is the True for every argument. Compare this with '$\begin{array}{l}\top\!\overset{\mathfrak{a}}{\smile}\!-\,\Phi(\mathfrak{a})\\ \llcorner\!-\!\!-\,\Gamma\end{array}$'. The latter refers to the False if Γ is the True and $\overset{\mathfrak{a}}{\smile}\!-\,\Phi(\mathfrak{a})$ is the False. But this is the case if for some argument the value of the function $-\!\!-\,\Phi(\xi)$ is the False. In all other cases $\begin{array}{l}\top\!\overset{\mathfrak{a}}{\smile}\!-\,\Phi(\mathfrak{a})\\ \llcorner\!-\!\!-\,\Gamma\end{array}$ is the True. The proposition '$\vdash\!\begin{array}{l}\top\!\overset{\mathfrak{a}}{\smile}\!-\,\Phi(\mathfrak{a})\\ \llcorner\!-\!\!-\,\Gamma\end{array}$' thus says the same as '$\vdash\!\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$'. If for '$\Gamma$' and '$\Phi(\xi)$' combinations of signs are put which do not refer to an object and a function but merely indicate by containing Roman letters, then the above still holds if for each Roman letter a name is put, whichever name it may be, and thus it holds generally.

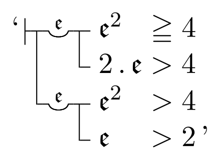
In order to be able to express myself more precisely, I will introduce the following terminology. I shall call *names* only those signs or combinations of signs that refer to something. Roman letters, and combinations of signs in which those occur, are thus not *names* as they merely *indicate*. A combination of signs which contains Roman letters, and which always results in
| a proper name when every Roman letter is replaced by a name, I will call 33a
a *Roman object-marker*. In addition, a combination of signs which contains Roman letters and which always results in a function-name when every Roman letter is replaced by a name, I will call a *Roman function-marker* or *Roman marker* of a function.

We can now say: the proposition '$\vdash\!\begin{array}{l}\top\!\overset{\mathfrak{a}}{\smile}\!-\,\Phi(\mathfrak{a})\\ \llcorner\!-\!\!-\,\Gamma\end{array}$' always says the same as the proposition '$\vdash\!\overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l}\top\,\Phi(\mathfrak{a})\\ \llcorner\,\Gamma\end{array}$' not only when '$\Phi(\xi)$' is a function-name and 'Γ' is a proper name, but also when '$\Phi(\xi)$' is a Roman function-marker and 'Γ' is a Roman object-marker.

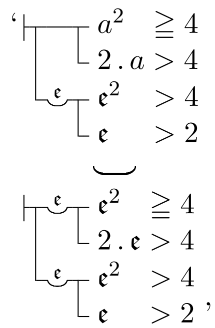
Let us apply this to the following example.



According to what has just been said, for the last proposition we can write instead



| It is clear that only those subcomponents that do not contain the Roman letter being replaced may be excluded from the scope of the German letter newly to be introduced. I will write such transitions like this: 33b



Instead of introducing several German letters one after the other, we will write the final result immediately under the sign '⏝'.

We will summarise this in the following rule:

> *A Roman letter may be replaced wherever it occurs in a proposition by one and the same German letter. At the same time, the latter has to be placed above a concavity in front of one such supercomponent outside of which the Roman letter does not occur.*[1] *If in this supercomponent the scope of a German let|ter* 34a

[1]The second rule of §8 does not prohibit here the repeated usage of '𝔢' since '*a*' does not occur in the first proposition within the scope of the '𝔢'.

[1]So, if the Roman letter occurs in every subcomponent, then the whole proposition excluding the judgement-stroke has to be regarded as the supercomponent, and the concavity with the German letter may be separated from the judgement-stroke only by a horizontal.

is contained and the Roman letter occurs within this scope, then the German letter that is to be introduced for the latter must be distinct from the former (second rule of §8).

§18. We will now set up some general laws for Roman letters which we will have to make use of later. According to §12

$$\begin{array}{l} \top\!\!\top\, \Gamma \\ \;\;\llcorner\, \Delta \\ \llcorner\, \Gamma \end{array}$$

would be the False only if Γ and Δ were the True while Γ was not the True. This is impossible; accordingly

$$\begin{array}{l} \vdash\!\top\!\!\top\, a \\ \quad\;\;\llcorner\, b \\ \quad\llcorner\, a \end{array} \qquad\qquad \text{(I}$$

‘I’ is given to this proposition as label (§14) and indices will be assigned in this manner to propositions in what follows. If we write ‘a’ instead of ‘b’ we can fuse equal subcomponents so as to obtain in ‘$\begin{array}{l}\vdash\!\top a \\ \;\;\llcorner\, a\end{array}$’ a special case | of 34b
(I), which is, without reminder, also to be understood as an instance of (I).

— Δ and $\top$ Δ are always different and are truth-values. Since — Γ is likewise always a truth-value, it must coincide with either — Δ or $\top$ Δ. From this it follows that $\begin{array}{l}\top (—\Gamma) = (—\Delta) \\ \llcorner\!\top (—\Gamma) = (\top\Delta)\end{array}$ is always the True; for it would only be the False if $\top$ (— Γ) = ($\top$ Δ) were the True, i.e., if (— Γ) = ($\top$ Δ) were the False, and (— Γ) = (— Δ) were not the True, i.e., the False. In other words: $\begin{array}{l}\top (—\Gamma) = (—\Delta) \\ \llcorner\!\top (—\Gamma) = (\top\Delta)\end{array}$ would only be the False if both (— Γ) = (— Δ) and (— Γ) = ($\top$ Δ) were the False, which as we have just seen is not possible. Therefore

$$\begin{array}{l} \vdash\!\top (—\, a) = (—\, b) \\ \;\;\llcorner\!\top (—\, a) = (\top\, b) \end{array} \qquad\qquad \text{(IV}$$

The brackets on the right side of the equality-sign can be dispensed with if one wishes.

From the reference of the function-name, $\backslash\xi$ (§11),

$$\vdash a = \backslash\dot{\varepsilon}(a = \varepsilon) \qquad\qquad \text{(VI}$$

follows.

| **§19.** So far, generality has been expressed only with respect to ob- 34a
jects. In order to be able to do the same for functions, we distinguish as *function-letters* the letters '*f*', '*g*','*h*','*F*','*G*','*H*' and the corresponding German letters, in contrast to the others that | we call *object-letters*,[1] so that 34b
they indicate only functions and never, like the latter, objects. We also count the small Greek | vowels amongst the *object-letters*, since their only 35a
occurrences without the smooth breathing are in places where a proper name may also stand. Within its scope, a function letter is always followed by a *bracket*, whose interior contains either one place, or two places separated by a comma, depending on whether the letter is to indicate a function with one or with two arguments. Such a place serves to receive a simple or complex sign which either refers to or indicates an argument or, like a small Greek vowel, occupies the argument place. It is clear that, within its scope, a function letter must always occur with one argument place or always with two argument places. In the case of a Roman function-letter, the *scope* comprises everything occurring in the proposition except the judgement-stroke; in the case of a German function-letter, it is demarcated by a concavity together with the German letter standing alone. In this regard, the use of function-letters conforms entirely with that of object-letters. To begin with, this may be illustrated by examples.

§20. $\smile^{\mathfrak{a}}\, \Phi(\mathfrak{a})$ is the True only if the value of the corresponding function $\Phi(\xi)$ is the True for every argument. So, $\Phi(\Gamma)$ likewise has to be the True. From this it follows that

$$\begin{array}{l} \top\!\!\!\!\!-\!\!\!-\, \Phi(\Gamma) \\ \quad \llcorner \smile^{\mathfrak{a}}\, \Phi(\mathfrak{a}) \end{array}$$

is always the True, whatever function with one argument $\Phi(\xi)$ may be. Here, in order to identify the corresponding function $\Phi(\xi)$, the | first rule of §8 is to be observed. 35b
If, e.g., one were to write '

$$\begin{array}{l} \top\!\!\!\!\!-\!\!\!-\, \Psi(\mathfrak{a}, \smile^{\mathfrak{a}}\, \mathrm{X}(\Gamma, \mathfrak{a})) \\ \quad \llcorner \smile^{\mathfrak{a}}\, \Psi(\mathfrak{a}, \smile^{\mathfrak{a}}\, \mathrm{X}(\mathfrak{a}, \mathfrak{a})) \end{array}$$

' one would only appear to have the same function named in both the super- and the subcomponent; in fact, the subcomponent would have been formed with the function-name '$\Psi(\xi, \smile^{\mathfrak{a}}\, \mathrm{X}(\mathfrak{a}, \mathfrak{a}))$' and the supercomponent with the function-name '$\Psi(\xi, \smile^{\mathfrak{a}}\, \mathrm{X}(\xi, \mathfrak{a}))$'. Now, '

$$\begin{array}{l} \smile^{\mathfrak{f}}\!\top\!\!\!\!\!-\!\!\!-\, \mathfrak{f}(\Gamma) \\ \quad\quad \llcorner \smile^{\mathfrak{a}}\, \mathfrak{f}(\mathfrak{a}) \end{array}$$

' is to be understood as the truth-value of: that a name of the True is always obtained no matter what function-name is inserted at the place of '$\mathfrak{f}$' into '

$$\begin{array}{l} \top\!\!\!\!\!-\!\!\!-\, \mathfrak{f}(\Gamma) \\ \quad \llcorner \smile^{\mathfrak{a}}\, \mathfrak{f}(\mathfrak{a}) \end{array}$$

'. This truth-value is the True whatever object is referred to by 'Γ':

$$\begin{array}{l} \vdash\!\smile^{\mathfrak{f}}\!\top\!\!\!\!\!-\!\!\!-\, \mathfrak{f}(a) \\ \quad\quad\quad \llcorner \smile^{\mathfrak{a}}\, \mathfrak{f}(\mathfrak{a}) \end{array}$$

Since the concavity with the '$\mathfrak{f}$' is only separated from the judgement-stroke by a horizontal, we can omit the concavity and write a Roman letter instead of the German

[1] With the exception of '*M*' which is reserved for a special purpose.

letter:

$$\vdash \begin{array}{l} f(a) \\ \smile^{\mathfrak{a}} f(\mathfrak{a}) \end{array} \qquad \text{(II a}$$

This law could be expressed in words like this: what holds of all objects, also holds of any.

According to §7, the function with two arguments $\xi = \zeta$ always has a truth-value as its value, namely the True if and only if the ζ-argument coincides with the ξ-argument. If $\Gamma = \Delta$ is the True, then so is | $\smile^{\mathfrak{f}} \begin{array}{l} \mathfrak{f}(\Gamma) \\ \mathfrak{f}(\Delta) \end{array}$; 36a
i.e., if Γ is the same as Δ, then Γ falls under every concept under which Δ falls, or, as one may also say, every predication that holds of Δ also holds of Γ. But also conversely: if $\Gamma = \Delta$ is the False, then not every predication that holds of Δ holds of Γ; i.e., in that case $\smile^{\mathfrak{f}} \begin{array}{l} \mathfrak{f}(\Gamma) \\ \mathfrak{f}(\Delta) \end{array}$ is the False. For example, Γ will not fall under the concept $\xi = \Delta$, under which Δ falls. Thus, $\Gamma = \Delta$ is always the same truth-value as $\smile^{\mathfrak{f}} \begin{array}{l} \mathfrak{f}(\Gamma) \\ \mathfrak{f}(\Delta) \end{array}$. Consequently, $\smile^{\mathfrak{f}} \begin{array}{l} \mathfrak{f}(\Gamma) \\ \mathfrak{f}(\Delta) \end{array}$ falls under every concept under which $\Gamma = \Delta$ falls; thus

$$\vdash \begin{array}{l} g\left(\smile^{\mathfrak{f}} \begin{array}{l} \mathfrak{f}(a) \\ \mathfrak{f}(b) \end{array}\right) \\ g(a = b) \end{array} \qquad \text{(III}$$

We saw in (§3, §9) that a value-range equality can always be converted into the generality of an equality, and *vice versa*:

$$\vdash (\dot{\varepsilon} f(\varepsilon) = \dot{\alpha} g(\alpha)) = (\smile^{\mathfrak{a}} f(\mathfrak{a}) = g(\mathfrak{a})) \qquad \text{(V}$$

In this, the first rules of §§8 and 9 are to be observed.

§21. In order to explain the use of function-letters in general, we require a further notational device that will now be explained.

Considering the names

$$\text{‘}\top\!\smile^{\mathfrak{a}}\!\top\, \mathfrak{a}^2 = 4\text{’} \qquad \text{‘}\top\!\smile^{\mathfrak{a}}\!\top\, \mathfrak{a} > 0\text{’}$$

$$\text{‘}\top\!\smile^{\mathfrak{a}}\!\top\top \begin{array}{l} \mathfrak{a}^2 = 1 \\ \mathfrak{a} > 0 \end{array}\text{’}$$

we can easily see that we obtain them | from ‘$\top\!\smile^{\mathfrak{a}}\!\top\, \Phi(\mathfrak{a})$’[1], by replacing the 36b
function-name ‘$\Phi(\xi)$’ with, respectively, the names of the functions, $\xi^2 = 4$, $\xi > 0$, $\top\top \begin{array}{l} \xi^2 = 1 \\ \xi > 0 \end{array}$. It is clear that only names of functions with one argument, not proper names or names of functions with two arguments, can be

[1] Compare §13.

inserted; for the combination of signs that is to be inserted always has to have open argument places to receive the letter '$\mathfrak{a}$'[2] and if we wanted to insert a name of a function with two arguments then the ζ-argument place would remain unfilled. In order, e.g., to insert the name of the function $\Psi(\xi, \zeta)$, one might be tempted to write '$\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\, \Psi(\mathfrak{a}, \mathfrak{a})$'; but, in truth, one would have inserted not the name of the function $\Psi(\xi, \zeta)$ but that of the function with one argument, $\Psi(\xi, \xi)$ (first rule of §8). If one were to write '$\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\, \Psi(\mathfrak{a}, 2)$' then one would again only insert the name of a function with one argument $\Psi(\xi, 2)$. One could, for example, leave the 'ζ' in place: '$\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\, \Psi(\mathfrak{a}, \zeta)$', and one would then have a function whose argument is indicated by 'ζ'. We take the foregoing into consideration along with the case where the argument-sign in '$\mathrm{X}(\xi)$' is replaced by '$\Phi(\xi)$': '$\mathrm{X}(\Phi(\xi))$'. It is common, yet inaccurate, to speak here of a | function of a function; for if we remind ourselves that 37a
functions are fundamentally different from objects and that the value of a function for an argument is to be distinguished from the function itself, then we can see that a function-name can never take the place of a proper name, because it will involve empty places corresponding to the unsaturatedness of the function. If we say 'the function $\Phi(\xi)$', then we should never forget that 'ξ' belongs with the function-name only by way of making the unsaturatedness recognisable. Another function thus can never occur as argument of the function $\mathrm{X}(\xi)$, although the value of a function for an argument can do so, for instance $\Phi(2)$, in which case the value is $\mathrm{X}(\Phi(2))$. If we write '$\mathrm{X}(\Phi(\xi))$', then by '$\Phi(\xi)$' we merely indicate an argument, just as 'ξ' in '$\mathrm{X}(\xi)$' merely indicates. The function-name is really just a part of '$\Phi(\xi)$' and so the function does not here occur as the argument of $\mathrm{X}(\xi)$, since the function-name only fills part of the argument place. Likewise, one cannot say that in '$\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\, \Psi(\mathfrak{a}, \zeta)$' the function-name $\Psi(\xi, \zeta)$ occupies the place of the function-name '$\Phi(\xi)$' in '$\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\, \Phi(\mathfrak{a})$'; since it fills only one part while the other, namely the place of 'ζ', remains open for a proper name. *Functions with two arguments* are just as fundamentally distinct from *functions with one argument* as the latter are from | *objects*. For, while the latter are fully 37b
saturated, functions with two arguments are less saturated than those with one argument, which are already *unsaturated*.

Thus, in '$\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\, \Phi(\mathfrak{a})$' we have an expression in which we can replace the name of the function $\Phi(\xi)$ by names of functions with one argument, but not by names of objects nor by names of functions with two arguments. This leads us to regard $\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\, \mathfrak{a}^2 = 4$, $\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\, \mathfrak{a} > 0$, $\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\!\top\!\!\begin{array}{l} \mathfrak{a}^2 = 1 \\ \llcorner\, \mathfrak{a} > 0 \end{array}$ as values of the same function $\top\!\stackrel{\mathfrak{a}}{\smile}\!\top\, \varphi(\mathfrak{a})$ with different *arguments*. However, here these arguments are again themselves functions, namely functions with one argu-

[2]That functions such as $\xi = \xi$ or $\xi^2 - \xi \,.\, \xi$ which have the same value for every argument — one may call them constant — must nevertheless be distinguished from this value (object) is shown in my lecture *Über Function und Begriff* (p. 8).

ment $\xi^2 = 4$, $\xi > 0$, $\begin{array}{l} \top\!\!\top\, \xi^2 = 1 \\ \llcorner\, \xi \;\; > 0 \end{array}$; and only functions with one argument can be arguments of our function $\top\!\overset{\mathfrak{a}}{\smile}\!\top\, \varphi(\mathfrak{a})$. If we say, 'the function $\top\!\overset{\mathfrak{a}}{\smile}\!\top\, \varphi(\mathfrak{a})$', then '$\varphi$' is proxy for the sign of an argument, just as 'ξ' in the expression 'the function $\xi^2 = 4$' is proxy for a proper name which could feature as argument-sign. In this case 'φ' no more belongs to the function than 'ξ' in the latter case. Functions whose arguments are objects we now call *first-level functions*; in contrast those functions whose arguments are first-level functions will be called *second-level functions.* The value of our function $\top\!\overset{\mathfrak{a}}{\smile}\!\top\, \varphi(\mathfrak{a})$ is always a truth-value whatever | first-level function we take as 38a
argument. In agreement with our earlier terminology, we will thus call it a concept, more precisely a *second-level concept*, in contrast to the *first-level concepts* which are first-level functions.

Our function $\top\!\overset{\mathfrak{a}}{\smile}\!\top\, \varphi(\mathfrak{a})$ had, for the arguments taken above, the True as its value. If we now take as its argument the function $\begin{array}{l} \top\!\!\top\, \xi^3 = -1 \\ \llcorner\, \xi \;\; > 0 \end{array}$, then we obtain in $\begin{array}{l} \top\!\overset{\mathfrak{a}}{\smile}\!\top\!\!\top\!\!\top\, \mathfrak{a}^3 = -1 \\ \qquad\;\; \llcorner\, \mathfrak{a} \;\; > 0 \end{array}$ the False, since there is no positive cubic root of -1. Likewise, the value of our function for the argument $\xi + 3$ is the False; for we can always replace $\top\!\overset{\mathfrak{a}}{\smile}\!\top\, (\mathfrak{a} + 3)$ by $\top\!\overset{\mathfrak{a}}{\smile}\!\top\, (\text{—}\, \mathfrak{a} + 3)$, and this is the False, since the value of the function $\text{—}\, \xi + 3$ is always the False, that is, if we assume that the plus-sign is explained in such a way that for no argument the value of the function $\xi + 3$ is the True.

§22. We have a different second-level function in

$$\begin{array}{l} \overset{\mathfrak{a}}{\smile}\!\top\!\overset{\mathfrak{e}}{\smile}\!\top\, \mathfrak{a} = \mathfrak{e} \\ \qquad\quad \llcorner\, \varphi(\mathfrak{e}) \\ \qquad \llcorner\!\!\text{——}\, \varphi(\mathfrak{a}) \end{array}$$

where 'φ' again is a proxy for the sign of the argument. Its value is the True for every first-level concept as argument under which at most a single object falls. Accordingly,

$$\begin{array}{l} \vdash\!\overset{\mathfrak{a}}{\smile}\!\top\!\overset{\mathfrak{e}}{\smile}\!\top\, \mathfrak{a} = \mathfrak{e} \\ \qquad\quad \llcorner\, \mathfrak{e} + 1 = 3 \\ \qquad \llcorner\!\!\text{——}\, \mathfrak{a} + 1 = 3 \end{array} \qquad \begin{array}{l} \vdash\!\overset{\mathfrak{a}}{\smile}\!\top\!\overset{\mathfrak{e}}{\smile}\!\top\, \mathfrak{a} = \mathfrak{e} \\ \qquad\quad \llcorner\!\top\, \mathfrak{e} = \mathfrak{e} \\ \qquad \llcorner\!\!\text{——}\!\top\, \mathfrak{a} = \mathfrak{a} \end{array}$$

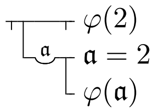
| In contrast: 38b

$$\begin{array}{l} \vdash\!\top\!\overset{\mathfrak{a}}{\smile}\!\top\!\overset{\mathfrak{e}}{\smile}\!\top\, \mathfrak{a} \;\; = \mathfrak{e} \\ \qquad\qquad \llcorner\, \mathfrak{e}^2 = 1 \\ \qquad\quad \llcorner\!\!\text{——}\, \mathfrak{a}^2 = 1 \end{array}$$

We also have a second-level function in $\varphi(2)$. Some of the values of this function are truth-values, as for example when the arguments are $\xi + \xi = \xi \,.\, \xi$, $\xi + 1 = 4$ to which correspond the values $2 + 2 = 2 \,.\, 2$ and $2 + 1 = 4$, while others are other objects, for instance the number 3 when the argument is $\xi + 1$. This second-level function is distinct from the mere number 2 since, like all functions, it is unsaturated.

The second-level function $—\varphi(2)$ is distinguished from the above by its value always being a truth value. It is thus a second-level concept which may be called property of the number 2; for every concept falls under this second-level concept under which the number 2 falls, while all other first-level functions with one argument do not fall under this second-level concept.[1]

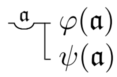
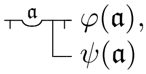
Also in



we have a second-level concept which we might call: property exclusively of the number 2.

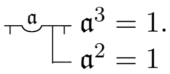
A further second-level concept is $\overset{\mathfrak{a}}{\smile}\, \varphi(\mathfrak{a})$. In $\grave{\varepsilon}\varphi(\varepsilon)$ we have an example of a second-level function which is not a concept.

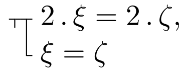
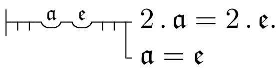
In order to take an example from analysis, | we will consider the dif- 39a
ferential quotient of a function. We regard the latter as argument. Let us take a specific function, e.g., ξ^2, as argument; then we first obtain another first-level function $2\,.\,\xi$ and only when we take as its argument an object, e.g., the number 3, do we obtain an object as value: the number 6. The differential quotient is therefore to be regarded as a function with two arguments, of which the one has to be a first-level function with one argument, and the other an object. We can therefore call it an *unequal-levelled* function with two arguments. From this we obtain a second-level function with one argument by saturating it with one object-argument — e.g., the number 3 — i.e., by determining that the differential quotient is to be formed for the argument 3.[1]

A further example of an unequal-levelled function with two arguments is provided by $—\varphi(\xi)$, where 'ξ' occupies the place of an object-argument and '$\varphi(\ )$' that of the function-argument and makes them salient. Since the value of this function is always a truth-value, we may call it an unequal-levelled relation. It is | the relation of an object to a concept under which it 39b
falls. Examples of equal-levelled relations of second-level are  and , where 'φ' and 'ψ' mark the argument places. The latter relation holds between, e.g., the concepts $\xi^3 = 1$ and $\xi^2 = 1$, because we have

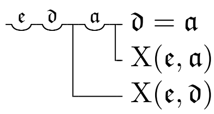
[1]Compare note p. 8.

[1]Here, as with all examples taken from arithmetic, it must be presupposed that the signs of addition, multiplication, and so on, as well as that of the differential quotient, are so defined that a name correctly formed from these and proper names always has a reference. The usual definitions certainly do not achieve this, for they invariably take into consideration only numbers, for the most part without saying what numbers are.

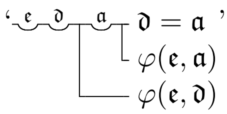
 In words: at least one square root of 1 is also a cube root of 1.

§23. In the examples given so far, functions with one argument featured as arguments; $\top\!\overset{\mathfrak{a}}{\smile}\!\overset{\mathfrak{e}}{\smile}\!\top\, \varphi(\mathfrak{a}, \mathfrak{e})$ is a second-level concept, whose argument has to be a function with two arguments. Under this concept falls every relation in which some objects stand. For one can also give relations — one might call them empty — in which no objects stand to one another; e.g.,  since 

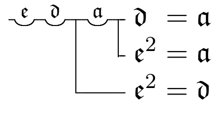
In order to have a further example of this kind, let us try to express the *single-valuedness* of a relation. By this we understand that for every ξ-argument there is no more than one ζ-argument such that the value of our function (relation) $\mathrm{X}(\xi, \zeta)$ is the True. We can also say: if, whenever a stands to b in this relation and a stands to c in | this relation, it follows 40a
that b and c coincide, then we say that the relation is single-valued. Or, if, whenever $\mathrm{X}(a, b)$ is the True and $\mathrm{X}(a, c)$ is the True, it follows that $c = b$ is the True, then we call the function $\mathrm{X}(\xi, \zeta)$ a single-valued relation, as long as it is a relation.



must be the True, if the relation $—\mathrm{X}(\xi, \zeta)$ is to be single-valued. If we introduce 'φ' to mark the argument place for 'X', then we obtain in



the name of a second-level function which requires as argument a function with two arguments. This second-level function is a second-level concept under which all single-valued relations fall, but also those functions $\mathrm{X}(\xi, \zeta)$ for which $—\mathrm{X}(\xi, \zeta)$ is a single-valued relation. The single-valuedness is always to be understood as running in the direction from the ξ to the ζ-argument. If we take the function $\xi^2 = \zeta$ as the argument of our second-level function, then we obtain the function-value



that is, the True; while the False is obtained as the value of the function if
| we take as the argument the function $\xi = \zeta^2$: 40b

$$\overset{\mathfrak{e}}{\smile}\,\overset{\mathfrak{d}}{\smile}\!\!\begin{array}{l} \overset{\mathfrak{a}}{\smile}\!\!\begin{array}{l} \mathfrak{d} = \mathfrak{a} \\ \llcorner\; \mathfrak{e} = \mathfrak{a}^2 \end{array} \\ \llcorner\!\!\longrightarrow \mathfrak{e} = \mathfrak{d}^2 \end{array}$$ [1]

These examples illustrate the great multiplicity of functions. We can also see that there are fundamentally different functions, since the argument places are fundamentally different. In particular, those which are suited to take proper names cannot receive names of functions, and conversely. Moreover, argument places that can take names of first-level functions with one argument are unable to take names of first-level functions with two arguments. Accordingly, we distinguish:

> *arguments of the first kind*:
> objects;
>
> *arguments of the second kind*:
> first-level functions with one argument;
>
> *arguments of the third kind*:
> first-level functions with two arguments.

Likewise, we distinguish:

> *argument places of the first kind*, that are suitable to take proper names;
>
> *argument places of the second kind*, that are suitable to take names of first-level functions with one argument;
>
> *argument places of the third kind*, that are suitable to take names
> | of first-level functions with two arguments. 41a

Proper names and object-letters *fit* the argument places of the first kind; names of first-level functions with one argument *fit* the argument places of the second kind; names of first-level functions with two arguments *fit* argument places of the third kind. The objects and functions whose names fit the argument places of names of functions are *fitting* arguments for these functions. Functions with one argument, for which arguments of the second kind are fitting, we call *second-level functions with an argument of the second kind*; functions with one argument, for which arguments of the third kind are fitting, we call *second-level functions with an argument of the third kind.*

Just as in $\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$ we have the value of the second-level function $\overset{\mathfrak{a}}{\smile}\, \varphi(\mathfrak{a})$ for the argument $\xi = \xi$, so too we can regard $\overset{\mathfrak{f}}{\smile}\!\!\begin{array}{l} \mathfrak{f}(1+1) \\ \llcorner\; \mathfrak{f}(2) \end{array}$ as the value of a *third-level* function for the argument $\begin{array}{l} \top\, \varphi(1+1) \\ \llcorner\; \varphi(2) \end{array}$ which is itself a second-level function with one argument of the second kind.

[1] Subject to a suitable definition of ξ^2 for arguments that are not numbers.

§24. It is now possible to give a general explanation of the use of function-letters.

If a concavity with a German function-letter | is followed by a combination of signs consisting of the name of a second-level function with one argument together with the function-letter in question in its argument places, then this whole refers to the True provided the value of that second-level function is the True for every fitting argument; in all other cases it refers to the False. Which places are argument places of the *corresponding* function is to be judged in accordance with the first rule of §8. The second rule of §8 also applies to function-letters, just as it applies to object-letters. 41b

We have hereby introduced two third-level functions, whose names can be displayed like this:

$$`\overset{\mathfrak{f}}{\smile}\, \mu_{\beta}(\mathfrak{f}(\beta))\text{'} \quad \text{and} \quad `\overset{\mathfrak{f}}{\smile}\, \mu_{\beta\gamma}(\mathfrak{f}(\beta,\gamma))\text{'},$$

where we mark the argument places by 'μ_{β}' and '$\mu_{\beta\gamma}$', just as we mark the argument places of the second and third kind by 'φ' and 'ψ', and those of the first kind by 'ξ' and 'ζ'. Like the former letters, incidentally, 'μ_{β}' and '$\mu_{\beta\gamma}$' are not to be regarded as signs of concept-script, but serve us only provisionally. If we take as arguments for the first of these functions the second-level functions with one argument of the second kind, $\overset{\mathfrak{a}}{\smile}\, \varphi(\mathfrak{a})$, $\varphi(2)$, $\overset{\mathfrak{a}}{\smile}\overset{\mathfrak{e}}{\smile}\begin{array}{l}\mathfrak{a}=\mathfrak{e}\\ \varphi(\mathfrak{e})\\ \varphi(\mathfrak{a})\end{array}$,

then we respectively obtain as value: $\overset{\mathfrak{f}}{\smile}\overset{\mathfrak{a}}{\smile}\, \mathfrak{f}(\mathfrak{a})$, $\overset{\mathfrak{f}}{\smile}\, \mathfrak{f}(2)$, and $\overset{\mathfrak{f}}{\smile}\overset{\mathfrak{a}}{\smile}\overset{\mathfrak{e}}{\smile}\begin{array}{l}\mathfrak{a}=\mathfrak{e}.\\ \mathfrak{f}(\mathfrak{e})\\ \mathfrak{f}(\mathfrak{a})\end{array}$

| **§25.** We still need a way to express generality with respect to second-level functions with one argument of the second kind. It might be thought that this would not nearly suffice; however we will see that we can get by with this expression and indeed that it is required only in one single proposition. Here it may merely be briefly remarked that this economy is made possible by the fact that second-level functions are representable, in a way, by first-level functions where the functions that appear as arguments of the former are represented by their the value-ranges. However the notational device that is necessary for this is not included in the primitives of concept-script; we will later introduce it by means of our primitive signs. Since this expressive device is only used in one single proposition, it is unnecessary to explain it in full generality. 42

We indicate a second-level function with one argument of the second kind by using the *Roman function-letter* 'M'[1] in this way:

$$`M_{\beta}(\varphi(\beta))\text{'}$$

[1]This letter is thus not an *object-letter.*

just as by '$f(\xi)$' we indicate a first-level function with one argument. Here, '$\varphi(\ )$' marks the argument place, just as 'ξ' in '$f(\xi)$'. The bracketed letter 'β' here fills the argument place of the function that occurs as argument. The use of '$M_\beta(\varphi(\beta))$' for second-level functions is completely analogous to that of $f(\xi)$ for first-level functions. We avail ourselves of this expression of generality in the following law:

$$\vdash\begin{array}{l} M_\beta(f(\beta)) \\ \overset{\mathfrak{f}}{\smile}\ M_\beta(\mathfrak{f}(\beta)) \end{array} \qquad \text{(II b}$$

in words: what holds of all first-level functions with one argument also holds of any. Obviously this law is to our second-level functions what (IIa) is to first-level functions. Here the letter 'f' in (II a) corresponds to 'M_β', the 'a' in (II a) to 'f', and the '$\mathfrak{a}$' to '$\mathfrak{f}$'. Let $\Omega_\beta(\varphi(\beta))$ be a second-level function with one argument of the second kind whose places are marked by 'φ'. Then $\overset{\mathfrak{f}}{\smile}\ \Omega_\beta(\mathfrak{f}(\beta))$ is the True only if the value of our second-level function is the True for every fitting argument. Then $\Omega_\beta(\Phi(\beta))$ also has to be the True. Thus,

$$\begin{array}{l} \Omega_\beta(\Phi(\beta)) \\ \overset{\mathfrak{f}}{\smile}\ \Omega_\beta(\mathfrak{f}(\beta)) \end{array}$$

is always the True, no matter what first-level function with one argument $\Phi(\xi)$ may be, regardless of whether $\overset{\mathfrak{f}}{\smile}\ \Omega_\beta(\mathfrak{f}(\beta))$ is the True or the False; and our law (II b) states this in general, for every second-level function with one argument of the second kind.

2. Definitions 43

General remarks

§26. The signs explained so far will now be used to introduce new names. But, before I discuss the rules that have to be followed here, it will promote comprehension to classify the signs and combinations of signs into kinds and to label them accordingly.[1]

I will not call the German, Roman and Greek letters occurring in concept-script *names* since they are not to refer to anything. In contrast, I do call, for example, '$\overset{\mathfrak{a}}{\smile}\, \mathfrak{a} = \mathfrak{a}$' a *name* since it refers to the True; it is a *proper name*. Thus, I call a *proper name* or *name* of an object any sign, be it simple or complex, that is to refer to an object, but not such a sign which merely indicates an object.

If we remove from a proper name some or all occurrences of another proper name that forms part of or coincides with it, but in such a way that these places remain marked as fillable by one and the same arbitrary proper name (as *argument places of the first kind*), then I call that which we obtain in this way a *name* of a first-level function with one argument. Such a name forms together with a proper name which fills the argument places a proper name. Accordingly, we also have in 'ξ' itself a function-name if the letter 'ξ' is merely to mark the argument place. The function so named has the property that its value for any argument coincides with the argument itself.

If we remove from a name of a first-level function with one argument all or some occurrences of a proper name that forms part of it, but in such a way that these places remain marked as fillable by one and the same arbitrary proper name (as argument places of the first kind), then I call that which we obtain in this way a *name* of a first-level function with two arguments.

If we remove from a proper name all or some occurrences of a name of a first-level function that forms part of it, but in such a way that these places remain marked as fillable by one and the same arbitrary name of a first-level function (as argument places of the second or third kind), then I call that which we obtain in this way a *name* of a second-level function | 44
with one argument, and specifically, of the second or third kind, depending on whether the argument places are of second or third kind.

Names of functions I call *function-names* for short.

It is not necessary to continue further this explanation of kinds of names.

If, in a proper name, we replace proper names that form part of or coincide with it by object-letters, function-names by function-letters, then I call that which we obtain in this way an *object-marker*, or *marker* of an object. If this replacement involves only Roman letters, then I call the marker obtained a *Roman object-marker*. Thus, object-letters are also object-markers and Roman object-letters are Roman object-markers.

[1] Compare §17.

A sign (proper name or object-marker) that only consists of the function-name ‘$\xi = \zeta$’ and proper names or object-markers occurring at the two argument places, I call an *equation*.

If, in a function-name, we replace proper names by object-letters, function-names by function-letters, then I call that which we obtain in this way a *function-marker*, specifically, a *marker* of a function of the same kind as the one from whose name it originates. If the replacement involves only Roman letters, then I call the marker obtained a *Roman* function-marker. Function-letters are also function-markers and Roman function-letters are Roman function-markers.

I reckon the judgement-stroke to belong neither with the *names* nor with the *markers*; it is a sign of its own kind. A sign which consists of a judgement-stroke and a name of a truth-value with a prefixed horizontal, I call a *concept-script proposition*, or *proposition*, where there can be no doubt. Likewise, I call a *concept-script proposition* (or *proposition*) a sign which consists of a judgement-stroke and a Roman marker of a truth-value with a prefixed horizontal.

Signs like

$$\text{‘}(\alpha) : \text{————’}, \quad \text{‘}(\alpha, \beta) :: = = = = =\text{’}, \quad \text{‘}(\alpha) :: - - - - -\text{’}, \quad \text{‘}\times\text{’}$$

that stand between propositions to indicate how the lower results from the one above, I call *transition-signs*.

§27. In order to introduce new signs by means of those already known, we now require the *double-stroke of definition* which appears as a double judgement-stroke combined with a horizontal:

$$\text{‘}\Vdash\text{’}$$

and which is used instead of the judgement-stroke where something is to be defined, rather than judged. By means of a *definition* we introduce a | new 45
name by determining that it is to have the same sense and the same reference as a name composed of already known signs. The new sign thereby becomes co-referential with the explaining sign; the definition thus immediately turns into a proposition. Accordingly, we are allowed to cite a definition just like a proposition replacing the definition-stroke by a judgement-stroke.

Here a definition is always presented in the form of an equation with a prefixed ‘$\Vdash$’. We will always write the explaining sign on the left side of the equality-sign, and the explained on the right side. The former will be composed of known signs.

§28. I now lay down the following governing principle for definitions:

Correctly formed names must always refer to something.

I call a name *correctly* formed if it consists only of such signs that are primitive or introduced by definition, and if these signs are only used as they were introduced to be used, that is, proper names as proper names, names of functions of first-level with one argument as names of such functions, and so on, so that the argument places are always filled with fitting names or markers. For *correct* formation it is further required, that German and small Greek letters are always used in accordance with their purpose. Thus a German letter may only stand above a concavity if the concavity is immediately followed by a marker of a truth-value composed of the name, or marker, of a function with one argument whose argument places are filled by the same German letter. Throughout its scope, a function-letter must occur everywhere either with one or with two argument places. A German letter may occur in an argument place only if a concavity with the same letter stands to the left of it, demarcating its scope. Only a German letter may stand above a concavity. A small Greek vowel may stand beneath a smooth breathing only when immediately followed by an object-marker consisting of a name or the marker of a first-level function with one argument with the same Greek vowel filling its argument places. A small Greek vowel may stand in an argument place only when preceded by the same vowel with a smooth breathing demarcating its scope. Under the smooth breathing only a small Greek vowel may occur.

§29. We now answer the question: when does a name refer to something? We confine ourselves to the following cases.

A name of a first-level function with one argument has | a *reference* 46
(*refers to* something, is *referential*) if the proper name which results from this function-name when the argument places are filled by a proper name always has a reference provided the inserted name refers to something.

A proper name has a *reference* if, whenever it fills the argument places of a referential name of a first-level function with one argument, the resulting proper name has a reference, and if the name of a first-level function with one argument which results from the relevant proper name's filling the ξ-argument places of a referential name of a first-level function with two arguments, always has a reference, and if the same also holds for the ζ-argument places.

A name of a first-level function with two arguments has a *reference* if the proper name always has a reference which results from this function-name by filling both the ξ-argument places with a referential proper name and the ζ-argument places with a referential proper name.

A name of a second-level function with one argument of the second kind has a *reference* if, whenever the name of a first-level function with one argument refers to something, it follows that the proper name that results by its insertion into the argument places of our second-level function has a reference.

Accordingly, every name of a first-level function with one argument which forms a referential proper name with every referential proper name, also forms a referential name with every referential name of a second-level function with one argument of the second kind.

The name '$\overset{\mathfrak{f}}{\smile}\,\mu_\beta(\mathfrak{f}(\beta))$' of a third-level function is referential if, whenever the name of a second-level function with one argument of the second kind refers to something, it follows that also the proper name that results by its insertion into the argument place of '$\overset{\mathfrak{f}}{\smile}\,\mu_\beta(\mathfrak{f}(\beta))$' has a reference.

§30. These propositions are not to be regarded as explanations of the expressions 'to have a reference' or 'to refer to something', since their application always presupposes that one has already recognised some names as referential; but they can serve to widen the circle of such names gradually. It follows from them that every name formed out of referential names refers to something. This formation takes place in such a way that a name fills argument places of another one that fit it. Thus a proper name results from a proper name and a name of a first-level function with one argument, or from a name of a first-level function and a name of a second-level function
| with one argument, or from a name of a second-level function with one 47
argument of the second kind and the name '$\overset{\mathfrak{f}}{\smile}\,\mu_\beta(\mathfrak{f}(\beta))$' of a third-level function. Thus a name of a first-level function with one argument results from a proper name and a name of a first-level function with two arguments. The names so formed can in turn be used in the same way for the formation of names, and all names resulting in this way are referential provided the primitive simple ones are.

A proper name can only come to be employed in this formation insofar as it fills the argument places of one of the simple or complex first-level functions. Complex names of first-level functions result in the manner described only from simple names of first-level functions with two arguments by a proper name's filling the ξ- or the ζ-argument place. The remaining argument places of the complex function-name are thus always also those of a simple name of a function with two argument places. From this it follows that a proper name that is part of a name so formed always stands, wherever it occurs, in an argument place of one of the simple names of first-level functions. If we now replace this proper name by another at some or all places, then the resulting proper name[a] is also formed in the manner described, so it too has a reference provided all simple names used therein are referential. To be sure, it is here presupposed that all simple names of first-level functions with one argument have only one argument place and that the simple names of first-level functions with two arguments have only one ξ- and one

[a] *Translators' Note*: Furth: function-name. Furth claims this has to be a typo (see his long footnote). He is wrong, but it is an understandable mistake (Heck/May/Wehmeier) — See Roy's appendix for detailed explanation and correction Furth's mistake.

ζ-argument place. If this were not the case, it could happen in the replacement just mentioned that related argument places of simple function-names are filled with different names, and in such a case an explanation of the reference would be lacking. However, this can always be avoided, and must be avoided in order to prevent the occurrence of names without reference. Indeed, it would serve no purpose to introduce several ξ-argument places and several ζ-argument places for the simple function-names. If we assume this, we can recognise the possibility of a second way to form names of first-level function. For we first form a name in the first way and then omit a proper name that forms a part of it (or coincides with it entirely) at some or all places, but in such a manner that the latter remain recognisable as argument places of the first kind. The function-name thereby resulting also always has a reference provided the simple names from which it is formed refer to something, and it can in turn be used to form referential names in the first or second way.

| Thus we can, e.g., form the function name '$\Delta = \zeta$' in the first way from 48
the proper name 'Δ' and the function name '$\xi = \zeta$'; and then the proper name '$\Delta = \Delta$' from this and 'Δ'. From this we can form in the second way the function-name '$\xi = \xi$', and from the latter and the function name '$\overset{\mathfrak{a}}{\smile}\ \varphi(\mathfrak{a})$', in the first way, the proper name '$\overset{\mathfrak{a}}{\smile}\ \mathfrak{a} = \mathfrak{a}$'.

All correctly formed names are so formed.

§31. We apply this to show that proper names and names of first-level functions that we can form in this manner from our previously introduced simple names, always have a reference. According to what has been said, it is required merely to demonstrate of our primitive names that they refer to something. These are

1. names of first-level functions with one argument:

$$\text{‘}{-\!\!-}\ \xi\text{’},\quad \text{‘}{\top}\ \xi\text{’},\quad \text{‘}\backslash\xi\text{’};$$

2. names of first-level functions with two arguments:

$$\text{‘}\begin{array}{l}\top\ \xi\\ \llcorner\ \zeta\end{array}\text{’},\quad \text{‘}\xi = \zeta\text{’};$$

3. names of second-level functions with one argument of the second kind:

$$\text{‘}\overset{\mathfrak{a}}{\smile}\ \varphi(\mathfrak{a})\text{’},\quad \text{‘}\dot{\varepsilon}\varphi(\varepsilon)\text{’};$$

4. names of third-level functions:

$$\text{‘}\overset{\mathfrak{f}}{\smile}\ \mu_{\beta}(\mathfrak{f}(\beta))\text{’},\quad \text{‘}\overset{\mathfrak{f}}{\smile}\ \mu_{\beta\gamma}(\mathfrak{f}(\beta,\gamma))\text{’}$$

of which the last may remain out of consideration since it will not be made use of.

First it may be noted that always only one ξ- and only one ζ-argument place occurs. We assume that the names of truth-values refer to something, namely either the True or the False. We will then gradually widen the circle of names to be recognised as referential, by demonstrating that the names that are to be added form referential names with those already added, by way of one occupying fitting argument places of the other.

Now, in order first to show that the function-names '$—\, \xi$' and '$\top\, \xi$' refer to something, we have to show that the names that result if we put a name of a truth-value for 'ξ' are referential (here, we are not yet recognising any other objects). This follows immediately from our explanations. The names obtained are again names of truth-values.

If we put names of truth-values for 'ξ' and 'ζ' in the function-names '$\begin{smallmatrix}\top\,\xi\\ \llcorner\,\zeta\end{smallmatrix}$' and '$\xi = \zeta$', then we obtain names that refer to truth-values. Consequently, our names of first-level functions with two arguments have references.

| In order to examine whether the name of a second-level function, 49
'$\smile^{\mathfrak{a}}\, \varphi(\mathfrak{a})$', refers to something, we ask whether, whenever a function-name '$\Phi(\xi)$' refers to something, it follows that '$\smile^{\mathfrak{a}}\, \Phi(\mathfrak{a})$' is referential. Now, '$\Phi(\xi)$' has a reference if, for every referential proper name 'Δ', '$\Phi(\Delta)$' refers to something. If so, this reference is either always the True (whatever 'Δ' may refer to), or not always. In the first case, '$\smile^{\mathfrak{a}}\, \Phi(\mathfrak{a})$' refers to the True, in the second, the False. Thus, whenever an inserted function-name '$\Phi(\xi)$' refers to something, it follows that '$\smile^{\mathfrak{a}}\, \Phi(\mathfrak{a})$' refers to something. Therefore, the function-name '$\smile^{\mathfrak{a}}\, \varphi(\mathfrak{a})$' is to be added to the circle of referential names. The same follows in a similar way for '$\smile^{\mathfrak{f}}\, \mu_{\beta}(\mathfrak{f}(\beta))$'.

The matter is less simple with '$\dot{\varepsilon}\varphi(\varepsilon)$'; for with this we introduce not only a new function-name but at the same time a new proper name (value-range name) for every name of a first-level function with one argument, and indeed not only for the ones already known, but also, in advance, for any that may yet be introduced. In investigating whether a value-range name refers to something, we need only consider those which are formed from referential names of first-level functions with one argument. For short, we will call them *regular*[a] value-range names. We must examine whether a regular value-range name that is put into the argument places of '$—\, \xi$' and '$\top\, \xi$' results in a referential proper name, and further whether, put in the ξ- or ζ-argument places of '$\begin{smallmatrix}\top\,\zeta\\ \llcorner\,\xi\end{smallmatrix}$' and '$\xi = \zeta$', it in each case forms a referential name of a first-level function with one argument. If we put the value-range name '$\dot{\varepsilon}\Phi(\varepsilon)$' for ζ in '$\xi = \zeta$', then the question becomes whether '$\xi = \dot{\varepsilon}\Phi(\varepsilon)$' is a referential name of a first-level function with one argument, and for this

[a] *recht* — see introduction

we need in turn to ask whether all proper names refer to something that result from putting either a name of a truth-value or a regular value-range name in the argument place. Owing to our stipulations that '$\grave{\varepsilon}\Psi(\varepsilon) = \grave{\varepsilon}\Phi(\varepsilon)$' is always to be co-referential with '$\smile^{\mathfrak{a}}\ \Psi(\mathfrak{a}) = \Phi(\mathfrak{a})$', that '$\grave{\varepsilon}(\text{—}\ \varepsilon)$' is to refer to the True and that '$\grave{\varepsilon}(\varepsilon = \top\!\smile^{\mathfrak{a}}\ \mathfrak{a} = \mathfrak{a})$' is to refer to the False, every proper name of the form '$\Gamma = \Delta$' is guaranteed a reference if 'Γ' and 'Δ' are regular value-range names or names of truth-values. Thereby it is also known that we always obtain a referential proper name from the function-name '$\xi = (\xi = \xi)$', if we put a regular value-range name in the argument places. Now, since according to our specifications the function $\text{—}\ \xi$ always has the same value for the same argument as the function $\xi = (\xi = \xi)$, then it is also known of the function-name '$\text{—}\ \xi$' that it always results in a proper name of a truth-value by insertion of a regular value-range name. | According 50
to our specifications the names '$\top\ \Delta$' and '$\begin{array}{l}\top\ \Gamma\\ \llcorner\ \Delta\end{array}$' always have references if the names '$\text{—}\ \Delta$' and '$\text{—}\ \Gamma$' refer to something. Now, since this is the case when 'Γ' and 'Δ' are regular value-range names, we always obtain from the function-names '$\top\ \xi$' and '$\begin{array}{l}\top\ \xi\\ \llcorner\ \zeta\end{array}$' referential proper names whenever we put regular value-range names or names of truth-values in the argument places. We have seen that each of our names of simple first-level functions hitherto recognised as referential, '$\text{—}\ \xi$', '$\top\ \xi$', '$\begin{array}{l}\top\ \xi\\ \llcorner\ \zeta\end{array}$', '$\xi = \zeta$', results in a referential name when we put regular value-range names in the argument places. The regular value-range names may thus be added to the circle of referential names. Thereby, however, the same is settled for our function-name '$\grave{\varepsilon}\varphi(\varepsilon)$', since whenever a name of a first-level function with one argument refers to something, it follows that the proper name that results by its insertion into '$\grave{\varepsilon}\varphi(\varepsilon)$' refers to something.

Of our primitive names only '$\backslash\xi$' now remains. We have specified that '$\backslash\Delta$' is to refer to Γ if 'Δ' is a name of a value-range $\grave{\varepsilon}(\varepsilon = \Gamma)$, that on the other hand '$\backslash\Delta$' is to refer to Δ if there is no object Γ such that 'Δ' is a name of the value-range $\grave{\varepsilon}(\varepsilon = \Gamma)$. In this way, a reference is secured in all cases for proper names of the form '$\backslash\Delta$' and therefore for the function-name '$\backslash\xi$'.

§32. Thus it is shown that our eight primitive names have a reference and thereby that the same holds of all names correctly formed out of them. However, not only a reference but also a sense belongs to all names correctly formed from our signs. Every such name of a truth-value *expresses* a sense, a *thought*. For owing to our stipulations, it is determined under which conditions it refers to the True. The sense of this name, the *thought*, is: that these conditions are fulfilled. Now, a concept-script proposition consists of a judgement stroke with a name, or a Roman marker, of a truth-value.

However, such a marker is transformed into the name of a truth-value when German letters are introduced for the Roman ones with concavities put in front, in accordance with §17. If we suppose this has been carried out, then there is only the case where the proposition is composed of the judgement stroke and a name of a truth-value. By mean of such a proposition it is then asserted that this name refers to the True. Now, since it at the same time expresses a thought, we have in every correctly formed concept-script proposition a judgement that a thought is true; and | so a thought cannot 51
be missing. It will be for the reader to clarify for himself the thought of each occurring concept-script proposition and I will strive to facilitate this as much as possible at the outset.

Now, the simple or complex names of which the name of a truth-value consists contribute to expressing the thought, and this contribution of the individual name is its *sense*. If a name is part of the name of a truth-value, then the sense of the former name is part of the thought expressed by the latter.

§33. The following principles govern the use of definitions.

1. Every name correctly formed from the defined names must have a reference. Thus, for each case it must be possible to supply a name, composed of our eight primitive names, that is co-referential with it, and, except for inessential choices of German and Roman letters, the latter must be unambiguously determined by the definitions.

2. From this it follows that the same must never be defined twice, since it would remain in doubt whether these definitions were in harmony with one another.

3. A defined name must be simple; i.e., it must not be composed of names known or still to be explained; for otherwise it would remain in doubt whether the explanations of the names are in accord with one another.

4. If on the left-hand side of a definitional equation we have a proper name that is correctly formed from our primitive names or defined names, then this will always have a reference and we can put on the right a simple, hitherto unused sign which is now introduced by our definition as a co-referential proper name, so that this sign may in future be replaced wherever it occurs by the name standing on the left. Evidently, it must never be used as a function-name, since the path back back to the primitive names would then be cut off.

5. A name that is introduced for a first-level function with one argument may only contain a single argument place. In the case of several argument places, it would be possible to fill these with different names, and then the defined name would be used as one of a function with several arguments, although not defined as such. In defining a name of a first-level function with one argument, the argument places on the left side of the definitional equation must be filled with the same Roman object-letter that marks the

argument place of the new function-name on the right. The definition then
states that the proper name resulting from | insertion of a referential proper 52
name into the argument place on the right, is always to be co-referential with the one which results from insertion of the same proper name into all argument places on the left. The single argument place of the explained name thus represents all those of the explaining name. Whenever the defined function-name occurs subsequently, its argument place must always be filled by a proper name or an object-marker.

6. A name that is introduced for a first-level function with two arguments must contain two and no more than two argument places. The mutually related argument places on the left must be occupied by one and the same Roman object-letter, which also marks one of the two argument places on the right; the non-related argument places must contain different Roman letters. The definition then states that the proper name resulting from insertion of referential proper names into the argument places on the right, is always to be co-referential with the one which results from insertion of the same proper names into the corresponding argument places on the left. The one argument place on the right represents all ξ-argument places on the left, the other, all ζ-argument places.

7. A Roman letter must accordingly never occur on one side of such a definitional equation which does not also occur on the other. If the object-marker on the left-hand side turns into a correctly formed proper name when Roman letters are replaced by proper names, then by our stipulations the explained function-name always has a reference.

Cases other than the foregoing will not appear in the sequel.

Special definitions

§34. It has already been observed in §25 that first-level functions can be used instead of second-level functions in what follows. This will now be shown. As was indicated, this is made possible by the fact that the functions appearing as arguments of second-level functions are represented by their value-ranges, although of course not in such a way that they simply concede their places to them, for that is impossible. In the first instance, our concern is only to designate the value of the function $\Phi(\xi)$ for the argument Δ, that is, $\Phi(\Delta)$, using 'Δ' and '$\acute{\varepsilon}\Phi(\varepsilon)$'. I do so in this way:

$$\text{`}\Delta \frown \acute{\varepsilon}\Phi(\varepsilon)\text{'}$$

which is to be co-referential with '$\Phi(\Delta)$'. The object $\Phi(\Delta)$ appears as the value of the function $\xi \frown \zeta$ with two arguments, Δ as the ξ-argument, and
$\acute{\varepsilon}\Phi\varepsilon$ as the ζ-argument. Next, however, we have to explain $\xi \frown \zeta$ for all | 53
possible objects as arguments. This can be done thus:

$$\Vdash \backslash\dot{\alpha}\left(\stackrel{\mathfrak{g}}{\smile}\!\!\begin{array}{l}\top\, \mathfrak{g}(a) = \alpha \\ \llcorner\, u = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\right) = a \frown u \qquad \text{(A}$$

Here, since a function with two arguments is being defined, two Roman letters occur both on the right and on the left. Although the explaining expression contains only known signs, some elucidation may not be superfluous. On the left, we have a Roman marker which results from the proper name '$\backslash\dot{\alpha}\left(\stackrel{\mathfrak{g}}{\smile}\!\!\begin{array}{l}\top\, \mathfrak{g}(\Theta) = \alpha \\ \llcorner\, \Gamma = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\right)$' by replacing '$\Theta$' by '$a$' and '$\Gamma$' by '$u$'.[a] This proper name has the form of $\backslash\dot{\alpha}\Phi(\alpha)$. In accordance with §11, two cases are to be distinguished here, depending on whether or not an object Δ can be supplied that is the only one falling under the concept $— \Phi(\xi)$. If so, Δ itself is $\backslash\dot{\alpha}\Phi(\alpha)$. Applied to the case at hand, this means that if there is an object Δ such that $\stackrel{\mathfrak{g}}{\smile}\!\!\begin{array}{l}\top\, \mathfrak{g}(\Theta) = \Delta \\ \llcorner\, \Gamma = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}$ is the True, while for every argument distinct from Δ the function $\stackrel{\mathfrak{g}}{\smile}\!\!\begin{array}{l}\top\, \mathfrak{g}(\Theta) = \xi \\ \llcorner\, \Gamma = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}$ has the False as value, then Δ itself is $\backslash\dot{\alpha}\left(\stackrel{\mathfrak{g}}{\smile}\!\!\begin{array}{l}\top\, \mathfrak{g}(\Theta) = \alpha \\ \llcorner\, \Gamma = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\right)$. Now, $\stackrel{\mathfrak{g}}{\smile}\!\!\begin{array}{l}\top\, \mathfrak{g}(\Theta) = \Delta \\ \llcorner\, \Gamma = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}$ is the True provided that there is a first-level function with one argument whose value is Δ for Θ as argument and whose value-range is Γ. Otherwise $\stackrel{\mathfrak{g}}{\smile}\!\!\begin{array}{l}\top\, \mathfrak{g}(\Theta) = \Delta \\ \llcorner\, \Gamma = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}$ is the False. If we assume that Γ is a value-range, then for any function whose value-range is Γ, it is determined by Γ what value the function has for the argument Θ. Then there is always one and only one such value, and this value is $\backslash\dot{\alpha}\left(\stackrel{\mathfrak{g}}{\smile}\!\!\begin{array}{l}\top\, \mathfrak{g}(\Theta) = \alpha \\ \llcorner\, \Gamma = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\right)$ or $\Theta \frown \Gamma$. If, however, Γ is not a value-range at all, then the function $\stackrel{\mathfrak{g}}{\smile}\!\!\begin{array}{l}\top\, \mathfrak{g}(\Theta) = \xi \\ \llcorner\, \Gamma = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}$ has the False as value for every argument, and then our stipulation is to be drawn upon that '$\backslash\Lambda$' is to refer to Λ itself if there is no object Δ such that Λ is the value-range $\dot{\varepsilon}(\Delta = \varepsilon)$. Thus, when Γ is not a value-range, '$\Theta \frown \Gamma$' refers to the value-range of the function whose value is the False for every argument, namely $\dot{\varepsilon}(\top\, \varepsilon = \varepsilon)$.

To summarise: two cases have to be distinguished in order for the value of the function $\xi \frown \zeta$ to be determined. When the ζ-argument is a value-range, then the value of the function $\xi \frown \zeta$ is the value of that function[b] whose value-range is the ζ-argument for the ξ-argument as argument. If on the other hand the ζ-argument is not a value-range, then the value of the function $\xi \frown \zeta$ is $\dot{\varepsilon}(\top\, \varepsilon = \varepsilon)$ for every ξ-argument.

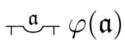
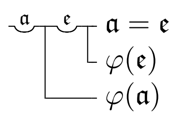
| **§35.** Here we see confirmed what we could gather from our earlier 54
considerations, namely, that the function-name '$\xi \frown \zeta$' has a reference. This

[a] Typo in Frege: quotation marks missing

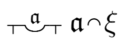
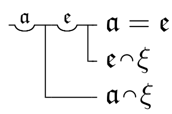
[b] *Translators' Note*: Scholz/Bachmann (see Thiel's corrigenda) wrongly suggest that it should be the value of *a* function. Explain what the sentence is about.

alone is fundamental for the conduct of the proofs to come; in other respects, our elucidation could be wrong without thereby calling into question the correctness of these proofs; for only the definition itself is the foundation for this construction. As mentioned at the outset, the goal was to enable us to use a first-level function instead of a second-level one. Let us now see how this goal is achieved. In §22 we introduced the second-level function $\varphi(2)$. Now, we can write '$2 \frown \dot{\varepsilon}\varphi(\varepsilon)$' for '$\varphi(2)$'. This is still the name of a second-level function; but if we write 'ξ' for '$\dot{\varepsilon}\varphi(\varepsilon)$', then we have in '$2 \frown \xi$' the name of a first-level function. The function $\varphi(2)$ has for the function $\Phi(\xi)$ as argument the same value, $\Phi(2)$, as has the function $2 \frown \xi$ for $\dot{\varepsilon}\Phi(\varepsilon)$ as argument. If an object is taken as argument of the function $2 \frown \xi$ that is not a value-range, then we have no corresponding argument for the second-level function $\varphi(2)$ and so the mutual representability of functions of the first and second levels lapses.

The second-level functions

 and 

correspond in the same manner to the first-level functions

 and 

§36. To find different examples, we seek to represent functions with two arguments by objects in a way similar to that followed for functions with one argument. A simple value-range cannot be used here, but only a double value-range, which is for a function with two arguments what the former is for a function with one argument.

For example, let us start with the function with two arguments $\xi + \zeta$. If we take, e.g., the number 3 as ξ-argument, then we have in $\xi + 3$ a function with just one argument, whose value-range is $\dot{\varepsilon}(\varepsilon + 3)$. The same holds for every ζ-argument, and we have in $\dot{\varepsilon}(\varepsilon + \zeta)$ a function with one argument, whose value is always a value-range. If we take the ξ- and the ζ-argument together with the value of the function $\xi + \zeta$ to be represented as rectangular co-ordinates in space, then we can display the value-range $\dot{\varepsilon}(\varepsilon + 3)$ as a straight line. If we allow the ζ-argument to vary continuously, then the straight line moves accordingly and thereby describes a plane. In each of its positions it displays a value-range, the value of the function $\dot{\varepsilon}(\varepsilon + \zeta)$ for a given ζ-argument. The value-range of the function $\dot{\varepsilon}(\varepsilon + \zeta)$ is now | 55a
$\dot{\alpha}\dot{\varepsilon}(\varepsilon + \alpha)$, and this is what I call a *double value-range.* Thus

$$\Delta \frown \dot{\alpha}\dot{\varepsilon}(\varepsilon + \alpha) = \dot{\varepsilon}(\varepsilon + \Delta)$$

is the True and so is

$$\Gamma\frown(\Delta\frown\dot{\alpha}\dot{\varepsilon}(\varepsilon+\alpha)) = \Gamma\frown\dot{\varepsilon}(\varepsilon+\Delta),$$

and since

$$\Gamma\frown\dot{\varepsilon}(\varepsilon+\Delta) = \Gamma+\Delta$$

is the True,

$$\Gamma\frown(\Delta\frown\dot{\alpha}\dot{\varepsilon}(\varepsilon+\alpha)) = \Gamma+\Delta$$

is also the True. Here on the left we see a double value-range representing a function with two arguments on the right, not however in such a way that the representing simply takes the place of the represented, which is impossible, but only in such a way that the double value-range on the left captures what differentiates the function on the right from other first-level functions with two-arguments. If a function with two arguments is a relation, then we may say '*extension* of the relation' as an alternative to 'double value-range'.

One can still ask what $\Gamma\frown(\Delta\frown\Theta)$ is when Θ is not a double value-range but merely a simple value-range or not a value-range at all. In the first case, $\Delta\frown\Theta$ is not a value-range and consequently $\Gamma\frown(\Delta\frown\Theta)$ is the same as $\dot{\varepsilon}(\top\, \varepsilon=\varepsilon)$. In the other case, $\Delta\frown\Theta$ coincides with $\dot{\varepsilon}(\top\, \varepsilon=\varepsilon)$, and

$$\Gamma\frown(\Delta\frown\Theta) = \Gamma\frown\dot{\varepsilon}(\top\, \varepsilon=\varepsilon)$$

is the True; therefore, $\Gamma\frown(\Delta\frown\Theta) = (\top\, \Gamma=\Gamma)$ is also the True; i.e., $\Gamma\frown(\Delta\frown\Theta)$ is then the False.

§37. Instead of the second-level function

$$\smile^{\mathfrak{e}}\smile^{\mathfrak{d}}\!\!\!\!\quad\smile^{\mathfrak{a}} \begin{array}{l} \mathfrak{d}=\mathfrak{a} \\ \varphi(\mathfrak{e},\mathfrak{a}) \\ \varphi(\mathfrak{e},\mathfrak{d}) \end{array}$$

| (§23) we can now consider the first-level function 55b

$$\smile^{\mathfrak{e}}\smile^{\mathfrak{d}}\quad\smile^{\mathfrak{a}} \begin{array}{l} \mathfrak{d}=\mathfrak{a} \\ \mathfrak{e}\frown(\mathfrak{a}\frown\xi) \\ \mathfrak{e}\frown(\mathfrak{d}\frown\xi)\,. \end{array}$$

We will introduce a simple notation for this by the following definition:

$$\Vdash\left(\smile^{\mathfrak{e}}\smile^{\mathfrak{d}}\quad\smile^{\mathfrak{a}} \begin{array}{l} \mathfrak{d}=\mathfrak{a} \\ \mathfrak{e}\frown(\mathfrak{a}\frown p) \\ \mathfrak{e}\frown(\mathfrak{d}\frown p) \end{array}\right) = \mathrm{I}p$$

(Γ

According to §23,

$$\smile^{\mathfrak{e}}\smile^{\mathfrak{d}}\quad\smile^{\mathfrak{a}} \begin{array}{l} \mathfrak{d}=\mathfrak{a} \\ \mathfrak{e}\frown(\mathfrak{a}\frown\Delta) \\ \mathfrak{e}\frown(\mathfrak{d}\frown\Delta) \end{array}$$

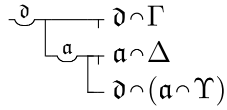
is the truth-value of: that the relation $—\xi\smallfrown(\zeta\smallfrown\Delta)$ is single-valued, i.e., that for every ξ-argument there is no or only one ζ-argument for which the value of the function is the True, or as we can also say, that for every object there is at most one to which it stands in the relation $—\xi\smallfrown(\zeta\smallfrown\Delta)$. If Δ is not a double value-range, then, according to §36, the value of the function $—\xi\smallfrown(\zeta\smallfrown\Delta)$[a] is either the False or $\dot{\varepsilon}(\top\,\varepsilon = \varepsilon)$. Since the latter is not the True, the value of the function $—\xi\smallfrown(\zeta\smallfrown\Delta)$ is thus always the False when Δ is not a double value-range; i.e., $—\xi\smallfrown(\zeta\smallfrown\Delta)$ is then a relation in which no object stands to any object. In that case, $\mathrm{I}\Delta$ is the True. The function-name '$\mathrm{I}\xi$' is introduced particularly in view of the cases in which an extension of a relation occurs as argument. If this relation is $\mathrm{X}(\xi,\zeta)$ then $\mathrm{I}\dot{\alpha}\dot{\varepsilon}\mathrm{X}(\varepsilon,\alpha)$ is the True when the relation $\mathrm{X}(\xi,\zeta)$ is single-valued (going | from the ξ- the 56a
ζ-argument). So, e.g., $\vdash \mathrm{I}\dot{\alpha}\dot{\varepsilon}(\varepsilon^2 = \alpha)$. According to our definitions, 'I' may be used only as a function-sign that precedes the argument-sign or its proxy.

§38. Now we can approach our objective of the definition of number. In my *Grundlagen der Arithmetik* I based this on a relation that I called equinumerosity. In §72 (p. 85) of my *Grundlagen*, I define:[b]

> The expression 'the concept F is equinumerous with the concept G' is co-referential with the expression 'there is a relation φ that is single-valued in both directions[c] and correlates the objects falling under the concept F with the objects falling under the concept G'.

What is it for a relation φ to correlate the objects falling under F with the objects falling under G? It is (§71 of *Grundlagen*) for every object that falls under F to stand in the relation φ with some object falling under G, or, more precisely, the two propositions 'a falls under F' and 'a stands to no object falling under G in the relation φ' cannot both hold for any a.

We now take $—\xi\smallfrown\Gamma$ as concept F, $—\xi\smallfrown\Delta$ as concept G, and $—\xi\smallfrown(\zeta\smallfrown\Upsilon)$ as the relation φ. Then we can express what is said in the symbolism of concept-script like this: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| The relation has to be single-valued. If we now add this, then we obtain 56b

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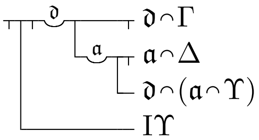
[a] Typo in Frege, noted by Thiel: '$—\xi\smallfrown(\xi\smallfrown\Delta)$'

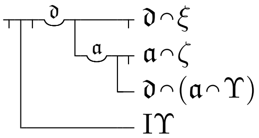
[b] Translators' Note containing Austin's translation of this passage from *Grundlagen*

[c] *beiderseits eindeutig* — re-consider: 'one-to-one'? — see introduction

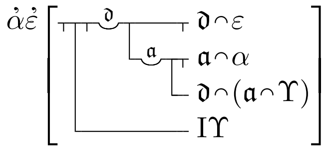
[1] The 'a' of the informal characterisation corresponds here to the '𝔡'.



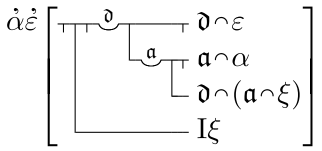
(Concerning ‘and’ compare §12). We consider this as the value of the function with two arguments



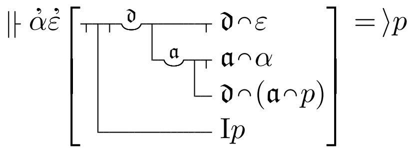
for the arguments Γ and Δ. This function is a relation. Its double-value-range is:



We consider the latter as the value of the function

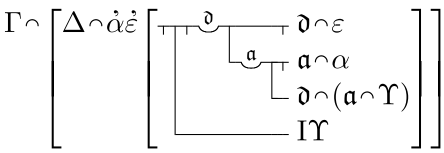


for the argument Υ. The following definition introduces a shortened name for this function:

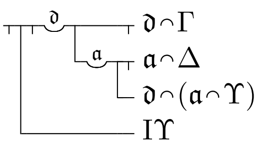


(Δ

The value of this function is always the extension of a relation. What accordingly is $\Gamma\frown(\Delta\frown\rangle\Upsilon)$? According to the definition we obtain the following:



| or 57a



This is the truth-value of: that the relation, $-\xi\frown(\zeta\frown\Upsilon)$, is single-valued and correlates the objects falling under the concept $-\xi\frown\Gamma$ with the objects falling under the concept $-\xi\frown\Delta$. For this, we will introduce the abbreviation, 'the Υ-relation *maps* the Γ-concept into the Δ-concept', calling, in general, a concept whose extension is Γ the *Γ-concept*, and a relation whose extension is Υ the *Υ-relation.*

§39. Now, if equinumerosity is to obtain between concepts, then there must be a relation of which holds not only what we just said of the Υ-relation but of whose converse also holds the same, when the roles of Γ and Δ are exchanged, so that it maps the Δ-concept into the Γ-concept. For this purpose, it is desirable to introduce a function-name '$\text{⅄}\xi$' in such a way that if Υ is the extension of a relation, then $\text{⅄}\Upsilon$ is the extension of its converse. For this purpose we define

$$\Vdash \dot{\alpha}\dot{\varepsilon}(\alpha\frown(\varepsilon\frown p)) = \text{⅄}p \qquad \text{(E}$$

The relation

$$-\xi\frown(\zeta\frown\text{⅄}\Upsilon) \quad \text{or}$$
$$-\xi\frown(\zeta\frown\dot{\alpha}\dot{\varepsilon}(\alpha\frown(\varepsilon\frown\Upsilon)))$$

is then the same as $-\zeta\frown(\xi\frown\Upsilon)$.

§40. Thus, in order to say the same of the converse of the relation
$-\xi\frown(\zeta\frown\Upsilon)$ as we have said of | the relation itself, we only need to replace 57b
'Υ' by '$\text{⅄}\Upsilon$'. Accordingly, $\begin{array}{l}\top\!\top\, \Gamma\frown(\Delta\frown\rangle\Upsilon)\\ \;\llcorner\, \Delta\frown(\Gamma\frown\rangle\text{⅄}\Upsilon)\end{array}$ is the truth value of: that the Υ-relation maps the Γ-concept into the Δ-concept and that its converse maps the latter into the former, assuming of course that Γ and Δ are extensions of concepts and Υ an extension of a relation. In order for these concepts to be equinumerous there has to be such a relation. $-\xi\frown(\zeta\frown\Upsilon)$ is always a relation, whatever object 'Υ' may refer to, and every relation may be designated in this form, '$-\xi\frown(\zeta\frown\Upsilon)$', by taking its extension for Υ. Accordingly, $\begin{array}{l}\top\!\overset{\mathfrak{q}}{\smile}\!\top\!\top\, \Gamma\frown(\Delta\frown\rangle\mathfrak{q})\\ \qquad\llcorner\, \Delta\frown(\Gamma\frown\rangle\text{⅄}\mathfrak{q})\end{array}$ is the truth-value of: that the concepts $-\xi\frown\Gamma$ and $-\xi\frown\Delta$ are equinumerous. We can regard this as the value of the function $\begin{array}{l}\top\!\overset{\mathfrak{q}}{\smile}\!\top\!\top\, \xi\frown(\Delta\frown\rangle\mathfrak{q})\\ \qquad\llcorner\, \Delta\frown(\xi\frown\rangle\text{⅄}\mathfrak{q})\end{array}$ for the argument Γ. This function is a concept whose extension is $\dot{\varepsilon}\left(\begin{array}{l}\top\!\overset{\mathfrak{q}}{\smile}\!\top\!\top\, \varepsilon\frown(\Delta\frown\rangle\mathfrak{q})\\ \qquad\llcorner\, \Delta\frown(\varepsilon\frown\rangle\text{⅄}\mathfrak{q})\end{array}\right)$. And, in accordance with my definition (*Grundlagen* §68), this extension of a concept is the *cardinal number* that belongs to the concept $-\xi\frown\Delta$. Instead of 'cardinal number that belongs to the Δ-concept', I also say for short 'cardinal number of the Δ-concept'. I now define:

$$\Vdash \dot{\varepsilon}\left(\overset{\mathfrak{q}}{\smile}\!\!\top\!\!\!\top \begin{array}{l} \varepsilon \frown (u \frown \rangle \mathfrak{q}) \\ u \frown (\varepsilon \frown \rangle \not\!\!\!\rangle \mathfrak{q}) \end{array} \right) = \wp u \qquad \text{(Z}$$

§41. Accordingly, $\wp\dot{\varepsilon}(\top\, \varepsilon = \varepsilon)$ is the cardinal number that belongs to the $\dot{\varepsilon}(\top\, \varepsilon = \varepsilon)$-concept, or the cardinal number that belongs to the concept, | $\top\, \xi = \xi$, and this is the cardinal number Zero (*Grundlagen* §74). Later, 58a
it will prove necessary to distinguish the cardinal number Zero from the number Zero,[a] and so I will mark the former with a slanting stroke. I define

$$\Vdash \wp\dot{\varepsilon}(\top\, \varepsilon = \varepsilon) = \emptyset \qquad (\Theta$$

§42. Likewise, I also define (*Grundlagen* §77)

$$\Vdash \wp\dot{\varepsilon}(\varepsilon = \emptyset) = \not{1} \qquad \text{(I}$$

The slanting stroke in '$\not{1}$' is meant to distinguish the cardinal number One from the number One. Accordingly, $\not{1}$ is the cardinal number that belongs to the concept $\xi = \emptyset$.

$\top\!\overset{\mathfrak{u}}{\smile}\!\top\, \wp\mathfrak{u} = \Gamma$ is the truth value of: that there is a concept to which the cardinal number Γ belongs or, as we can also say, that Γ is a cardinal number. Therefore, we call the function, $\top\!\overset{\mathfrak{u}}{\smile}\!\top\, \wp\mathfrak{u} = \xi$, the concept *cardinal number.*

§43. The relation in which one member of the cardinal number series stands to that immediately following it still remains to be explained. I will here give my definition (*Grundlagen* §76) in a slightly modified formulation:

> If there is a concept, $— \xi \frown \Gamma$, and an object Δ falling under it, such that the cardinal number belonging to the concept, $— \xi \frown \Gamma$, is Λ and the cardinal number belonging to the concept, $\top\!\!\!\top \begin{array}{l} \xi = \Delta, \\ \xi \frown \Gamma \end{array}$ is Θ, then I say: Λ follows immediately after Θ in the cardinal number series.

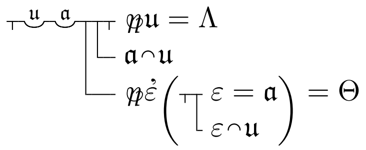
We now have in

$$\top\!\!\!\top\!\!\!\top \begin{array}{l} \wp\Gamma = \Lambda \\ \Delta \frown \Gamma \\ \wp\dot{\varepsilon}\left(\top\!\!\!\top \begin{array}{l} \varepsilon = \Delta \\ \varepsilon \frown \Gamma \end{array} \right) = \Theta \end{array}$$

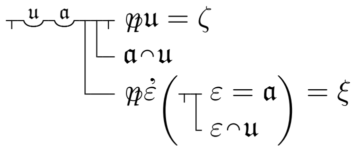
| the truth value of: that Λ is the cardinal number that belongs to the 58b
concept, $— \xi \frown \Gamma$, that Δ falls under this concept, and that Θ is the cardinal

[a] *Translators' Note*: "*Zahl Null*". — Compare Furth's Introduction, p. liv, where he quotes GG vol. 2, p. 155–156.

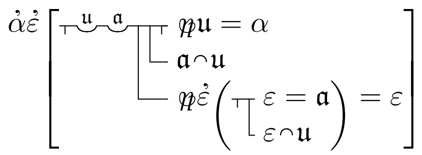
number of the $\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = \Delta\\ \llcorner\, \varepsilon \frown \Gamma\end{array}\right)$-concept. Accordingly, we have in



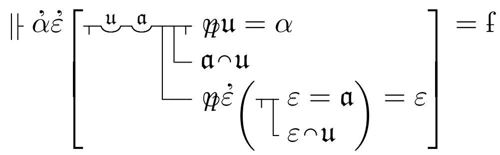
the truth-value of: that Λ follows immediately after Θ in the *cardinal number series*. We regard this as the value of the function



for the arguments Θ and Λ. The extension of this relation is



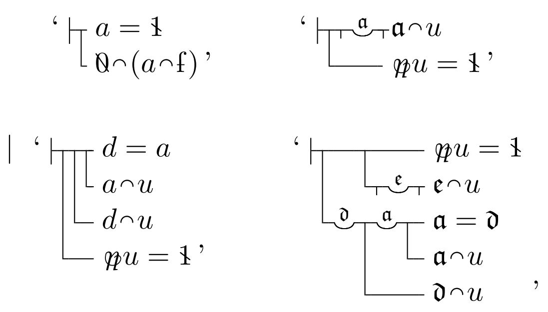
and for this a simple name will be introduced:



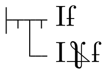
(H

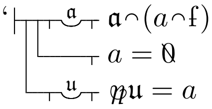
Accordingly, ‘$0 \frown (1 \frown \mathfrak{f})$’ expresses that 1 follows immediately after 0 in the cardinal number series.

§44. The six propositions listed in §78 of my *Grundlagen* may now be expressed in our symbolism as follows:



59a

I leave it to the reader to make the sense clear to himself. ‘$\mathrm{I}\mathfrak{f}$’ expresses that the $\mathfrak{f}$-relation is single-valued, in other words: that for every cardinal number there is no more than a single one which follows it immediately in the cardinal number series. And ‘I⅄𝔣’ expresses that for every cardinal number there is no more than a single one which it immediately follows. By ‘’ the fifth of the propositions is rendered.

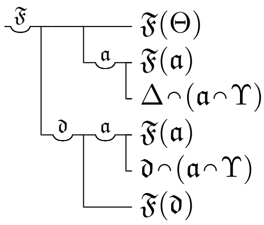
‘’

says that for every cardinal number, with the exception of Ø, there is one that immediately precedes it in the cardinal number series.

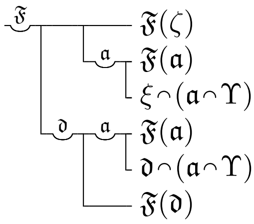
§45. The 𝔣-relation orders the cardinal numbers in such a way that a series results. We now have to explain in general what is stated by ‘an object follows after an object in a series’, where the type of this series is determined by the relation in which a member of the series always stands to the one following. I repeat in slightly different words the explanation that I gave in §79 in my *Grundlagen* and in the *Begriffsschrift*.

> If the proposition
>
> ‘If every object to which Δ stands in the Υ-relation falls under the concept — $F(\xi)$ | and if, whenever an object falls under this concept, it follows that every object to which it stands in the Υ-relation also falls under the concept — $F(\xi)$ then Θ falls under this concept’ 59b
>
> holds generally for every concept — $F(\xi)$, then we say:
>
> *Θ follows in the Υ-series after Δ.*

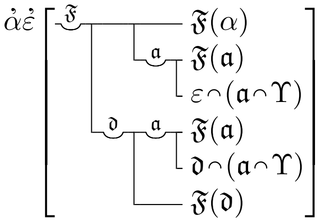
Accordingly,



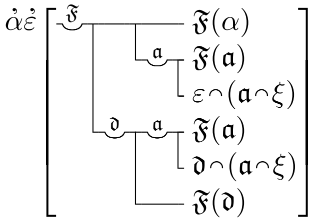
is the truth-value of: that Θ follows in the Υ-series after Δ. We can also regard this as the value of the function



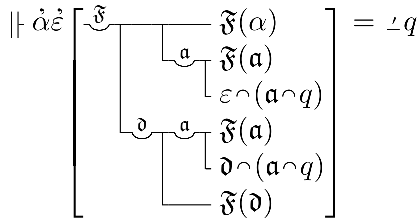
for Δ and Θ as arguments. The extension of this relation is



We can regard it as the value of the function



| for Υ as argument. For this function I introduce a simple name by defining: 60a



(K

Accordingly, '$\Delta\frown(\Theta\frown\perp\Upsilon)$' expresses that Θ follows after Δ in the Υ-series. And '$\Delta\frown(\Theta\frown\perp\mathrm{f})$' expresses that Θ follows after Δ in the cardinal number series. Instead of 'Θ follows after Δ in the Υ-series' I also say '*Δ precedes* Θ *in the Υ-series'*.

§46. $\begin{array}{l}\Theta = \Delta \\ \Delta\frown(\Theta\frown\perp\Upsilon)\end{array}$ is the truth-value of: that Θ either follows after Δ in the Υ-series or coincides with Δ. For short, I say that *Θ belongs to the Υ-series starting with Δ*, or that *Δ belongs to the Υ-series ending with Θ*. I regard this as the value of the function $\begin{array}{l}\zeta = \xi \\ \xi \frown (\zeta\frown\perp\Upsilon)\end{array}$ for Δ and Θ as arguments. The extension of this relation is $\dot{\alpha}\dot{\varepsilon}\left(\begin{array}{l}\alpha = \varepsilon \\ \varepsilon \frown (\alpha\frown\perp\Upsilon)\end{array}\right)$. I regard it as the value of the function

$$\dot{\alpha}\dot{\varepsilon}\left(\begin{array}{l}\alpha = \varepsilon \\ \varepsilon \frown (\alpha\frown\perp\xi)\end{array}\right)$$

for Υ as argument, and I introduce a simple name by defining:

$$\Vdash \dot{\alpha}\dot{\varepsilon}\left(\begin{array}{l} \top\!\!-\, \alpha = \varepsilon \\ \llcorner_{\top}\, \varepsilon \frown (\alpha \frown \underline{\smile} q) \end{array}\right) = \smile q \qquad (\Lambda$$

Accordingly, $\Delta \frown (\Theta \frown \smile \Upsilon)$ is the truth-value of: that Θ belongs to the Υ-series starting with Δ. Thus, $\mathfrak{0} \frown (\Theta \frown \smile \Upsilon)$ is the truth-value of: that Θ belongs to the cardinal number series starting with $\mathfrak{0}$, for which I can also say that Θ is a *finite* cardinal number.

In §82 of my *Grundlagen*, I mentioned the proposition that the cardinal number that belongs to the concept

belonging to the cardinal number series ending with n

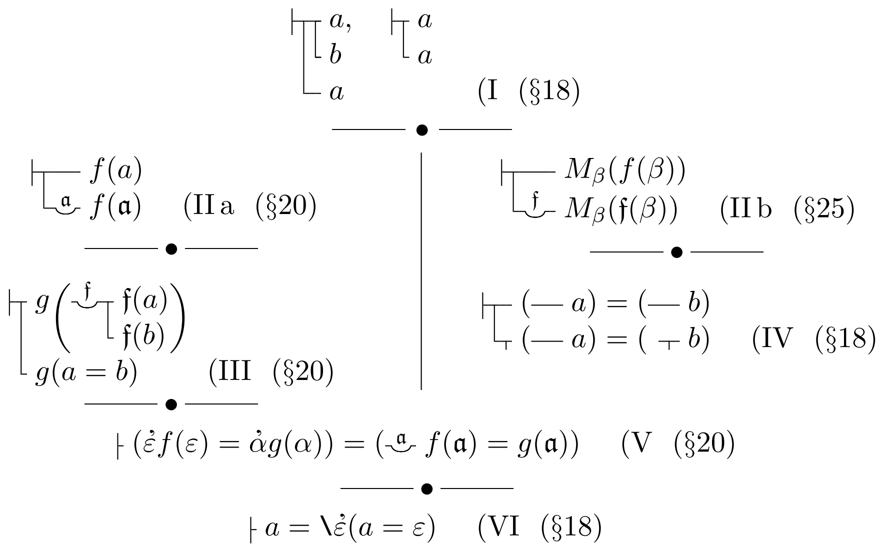
follows n immediately in the cardinal number series if n is a finite cardinal number. This can now be represented thus:

$$\text{‘}\begin{array}{l} \vdash\!\top\, n \frown (\wp(n \frown \smile \mathfrak{f}) \frown \mathfrak{f}) \\ \;\;\llcorner\, \mathfrak{0} \frown (n \frown \smile \mathfrak{f}) \end{array}\text{’;}$$

since $(\Theta \frown \smile \mathfrak{f})$ is the value of the concept, *belonging to the cardinal number series ending with Θ*.

3. Derived laws

§47. We have been seeing how concepts and objects with which we will be occupied later can be designated with our signs. This would be of little significance, however, if one could not also calculate with them, if | series of 60b
inferences could not be represented without mixing in words, proofs not be conducted. We have now become familiar with the basic laws and modes of inference that are going to be employed for this purpose. We will now derive laws from them for later use, | in such a way that the method of calculating 61a
is illustrated at the same time. I will first summarise the basic laws and rules, | and add some supplementary points. 61b



§48. *Summary of the rules* | 61a

1. *Fusion of horizontals*

If as argument of the function $—\,\xi$ there occurs the value of this same function for some argument, then the horizontals may be fused.

The two parts of the horizontal line that are separated by the negation stroke in '$\top\,\xi$' are horizontals in our sense.

The lower and the two parts of the upper horizontal stroke in '$\begin{array}{l}\top\,\xi\\ \llcorner\,\zeta\end{array}$' are horizontals in our sense.

Finally, the two straight strokes which are adjoined to the con|cavity in '$\smile^{\mathfrak{a}}\,\varphi(\mathfrak{a})$' are horizontals in our sense. 61b

2. *Permutation of subcomponents*

Subcomponents of the same proposition may be permuted with one another arbitrarily.

3. *Contraposition*

A subcomponent in a proposition may be permuted with a supercomponent provided one also inverts their truth values.[a]

Transition-sign: $\times$

[a] *Translators' Note*: For the remainder of this section, we have omitted periods and semicolons occurring in the original text adjacent to formulae and transition-signs.

4. *Fusion of equal subcomponents*

A subcomponent that occurs repeatedly in the same proposition only needs to be written once.

| 5. *Transformation of a Roman letter into a German letter* 62a

A Roman letter may be replaced wherever it occurs in a proposition by one and the same German letter, namely, an object-letter by an object-letter and a function-letter by a function-letter. At the same time, the latter has to be placed above a concavity in front of one such supercomponent outside of which the Roman letter does not occur. If in this supercomponent the scope of a German letter is wholly contained and the Roman letter occurs within this scope, then the German letter that is to be introduced for the latter must be distinct from the former.

Transition-sign: $\smile$

This sign is also used if several German letters are to be introduced in this way. Although one may write down the final product straightaway, one must think of them as introduced one after another.

6. *Inferring* (*a*)

If a subcomponent of a proposition differs from another proposition only in lacking the judgement-stroke, then one may infer a proposition which results from the first by suppressing that subcomponent.

Transition-signs: ( ) : ———

and ( ) :: ———

Combined inferences thus:

(,) :: ═══

| 7. *Inferring* (*b*) 62b

If the same combination of signs (either a proper name or a Roman object-marker) occurs in one proposition as supercomponent and in another as subcomponent, then a proposition may be inferred in which the supercomponent of the second proposition features as supercomponent and all subcomponents of both, save that mentioned, feature as subcomponents. In this, equal subcomponents may be fused according to rule (4).

Transition-signs: ( ) : – – –

and ( ) :: – – –

Combined inferences thus:

(,) :: = = = = and (,) :: ════

8. *Inferring (c)*

If two propositions agree in their supercomponents while a subcomponent of the one differs from a subcomponent of the other only by a prefixed negation stroke, then we can infer a proposition in which the common supercomponent features as supercomponent, and all subcomponents of both propositions with the exception of the two mentioned feature as subcomponents.

Transition-sign: ( ) : ·—·—·—·—·

9. *Citing propositions: replacement of Roman letters*

When citing a proposition by its label, we may effect a simple inference by uniformly replacing a Roman letter within the proposition by the same proper name or the same Roman object-marker.

| Likewise, we may replace all occurrences in a proposition of a Roman 63a
function-letter, 'f', 'g', 'h', 'F', 'G', 'H', by the same name or Roman marker of a first-level function with one or two arguments, depending on whether the Roman letter indicates a function with one or two arguments.

When we cite law (IIb) we may replace both occurrences of 'M_β' by the same name or Roman marker of a second-level function with one argument of the second kind.

Concerning the word 'same'[a] in the second and third paragraphs of this rule, it is to be observed that the argument is not a part of the function; that is, that a change of the argument-sign is not a change of the function-name. In order for the same function-name to occur in different locations, it is required that the related argument places appropriately match. Concerning the question which argument places are to be regarded as related, these rules are to be observed:

All occurrences of a German letter within its scope, save those within an enclosed scope of the same letter or above a concavity, are related argument places of the corresponding function;

All occurrences of a small Greek vowel within its scope, save those within
an enclosed scope of the same | letter or together with the smooth breathing, 63b
are related argument places of the corresponding function.

10. *Citing propositions: replacement of German letters*

When citing a proposition by its label, we may uniformly replace a German letter above a concavity and at all argument places of the corresponding function by one and the same distinct letter, that is, an object-letter by an object-letter and a function-letter by a function-letter, just provided no Ger-

[a] *Translators' note*: "*Im Betreff der Wörter* '*denselben*' *und* '*dieselbe*' [...]" – the gender difference of the German words is lost in the translation.

man letter occurring in a scope enclosed within its own thereby becomes the same as the one whose scope is enclosed.

11. *Citing propositions: replacement of Greek vowels*

When citing a proposition by its label, we may uniformly replace a Greek vowel under a smooth breathing and at all argument places of the corresponding function by one and the same distinct letter, just provided no Greek letter occurring in a scope enclosed within its own thereby becomes the same as the one whose scope is enclosed.

12. *Citing definitions*

When citing a definition by its label, we may replace the definition stroke
by the | judgement stroke and make those modifications that are allowed 64a
according to (9), (10), (11) when citing a proposition.

Stipulations concerning the use of brackets

13. Everything standing to the right of a horizontal, combined in a context, is to be considered as a whole that stands in the place of 'ξ' in '— ξ', provided brackets do not forbid it.

14. Everything standing to the left of an equality-sign, combined in a context, up to but excluding the nearest horizontal, is to be considered as a whole that stands in place of 'ξ' in '$\xi = \zeta$', provided brackets do not forbid it.

Accordingly, '$a = b = c$', e.g., is to be understood as '$(a = b) = c$'. Since, however, '$a = b = c$' is commonly used in a different sense, I will in such a case write down the brackets.

15. Everything standing to the right of an equality-sign up to but excluding the nearest equality-sign, is to be considered as a whole that stands in place of 'ζ' in '$\xi = \zeta$', provided brackets do not forbid it.

16. We have names of functions with two arguments, e.g., '$\xi = \zeta$', '$\xi \frown \zeta$', which have argument places both to the left and to the right. I will call such function-signs *two-sided*. For two-sided function-signs, except the equality-sign, the following is specified.

Everything standing to the left of such | a sign, combined in a context, 64b
up to the nearest equality-sign or horizontal, is to be considered as a whole that stands at the left argument place, provided brackets do not forbid it, and everything standing to the right of such a sign, combined in a context, up to the nearest two-sided function-sign, is to be considered as a whole that stands at the right argument place, provided brackets do not forbid it.

17. So far, simple names of first-level functions with one argument have been, and will continue to be, formed so that the argument place stands

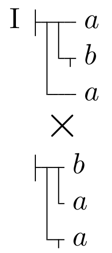
to the right of the function-sign proper, as in ‘Iξ’, ‘⟩ξ’, ‘⅄ξ’, ‘℘ξ’, ‘$\acute{-}\xi$’, ‘⌣ξ’. For such *one-sided* function-signs, except the horizontal, I specify the following.

Everything standing to the right of a one-sided function-sign, combined in a context, up to the nearest two-sided function-sign, is to be considered as a whole that stands at the argument place.

18. If a horizontal ends free on the left, then we enclose it together with its argument-sign in brackets.

§49. Let us first derive some propositions from (I).

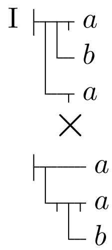
I will now cite (I) in such a way that I will write ‘⊤ *b*’ for ‘*b*’ by rule (9) of §48, and fuse the horizontals by rule (1). What follows will illustrate how a proposition is cited. | 65a



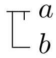
(I a

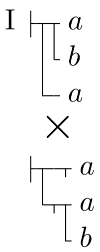
(I a) is hereby made the label of the new proposition. Concerning the transition, compare rule (3). Here we also made use of the permutability of the subcomponents according to rule (2).

In the following derivation, I cite (I) in a way that involves writing ‘⊤ *a*’ for ‘*a*’.

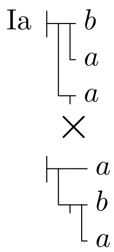


(I b

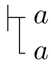
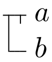
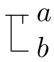
In the application of rule (3), ‘  ’ is to be regarded as the supercomponent.

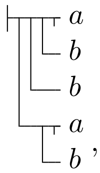


(I c

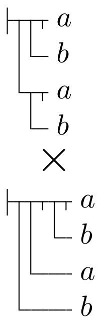


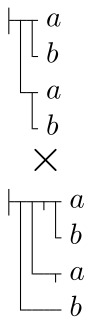
(I d

In the following derivation, (I) is to be thought of in the form, ‘’, where ‘’ is now written in place of ‘a’. If we assume (I) in its original form and write ‘’ for ‘a’, then initially we obtain: | 65b

‘’

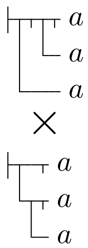
where the equal subcomponents may be fused according to rule (4). The following can also be understood in this way.

I  (I e)

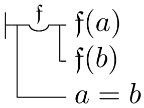
I  (I f)

Compare here what was said about ‘and’ in §12.

In the following (I e) is cited in a way that involves writing ‘a’ for ‘b’ and fusing the equal subcomponents:

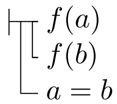
I e  (I g)

§50. Now the principal laws for the function $\xi = \zeta$ will be derived. First, by rule (9) §48, we replace the function-letter ‘g’ in (III) by the name of the function —— ξ and fuse the horizontals.

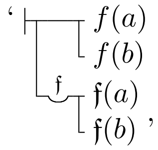
III 

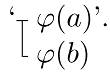
(II b): – – ––– –

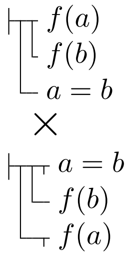
| 66a

III  (III a

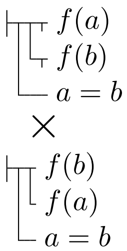
The transition here is made by rule (7), and (II b) is cited in the form

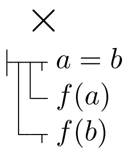


in which, by rule (9), ‘$M_\beta(\varphi(\beta))$’ is replaced by the Roman marker of a second-level function .

III a  (III b

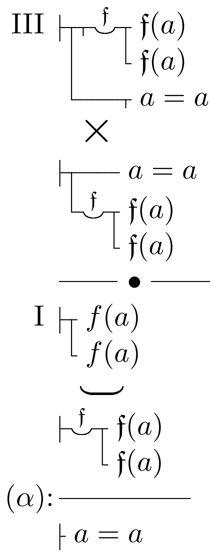
In the following derivation the function-letter ‘f’ in (III a) is replaced by the Roman function-marker ‘$\top f(\xi)$’, and then the horizontals are fused.

III a  (III c

 (III d

We can render (III a) in words roughly like this: if a and b coincide, then everything that | holds of b holds of a. Similarly, (III c). (III d) may be 66b
paraphrased like this: if a predication holds of a that does not hold of b, then a and b do not coincide.

In the following derivation the function-letter ‘g’ in (III) is replaced by the function-name ‘$\top \xi$’, and ‘b’ by ‘a’.



(α

(β

(III e

In the light of our explanation of the equality-sign, this proposition is indeed self-evident, but it is worth the effort to see how it can be established from (III). Further, doing so provides an opportunity to note some points that are also to hold for later derivations. This second proposition received the label ‘α’. When so used, a small Greek letter is only required to remain fixed as a label within the same derivation, and so may be used to label a different proposition in a different derivation. A derivation terminates in a proposition to which a label other than a small Greek let|ter is first assigned. 67a
In our derivation the proposition (α) is followed by the sign

‘—— ● ——’

to note that we break a series of inferences here and start a new one which is only connected to the former where we cite (α). The transition to (β) is made by rule (5), that from (β) to (III e) by rule (6). Let us now replace the function-letter ‘f’ in (III a) by the Roman function-marker, ‘$b = \xi$’:

III a $\vdash$ $b = a$
$b = b$
$a = b$

(III e)::

$\vdash$ $b = a$
$a = b$ (III f

This inference is made by rule (6).

In the following derivation, 'a' is replaced by '— a', 'b' by '$\top$ a', and the function-letter 'f' by the function-name '— ξ' in (III c) and (III a), and the horizontals are fused where possible.

III $\vdash$ a
a
$(— a) = (\top a)$

$\times$

$\vdash$ $(— a) = (\top a)$
a (α

—— $\bullet$ ——

III a $\vdash$ a
a
$(— a) = (\top a)$

$\times$

$\vdash$ $(— a) = (\top a)$
a (β

(α): · — · — · — ·

$\vdash$ $(— a) = (\top a)$ (III g

| In the transitions to (α) and (β) equal subcomponents are fused in accordance with rule (4). The final inference is made by rule (8). 67b

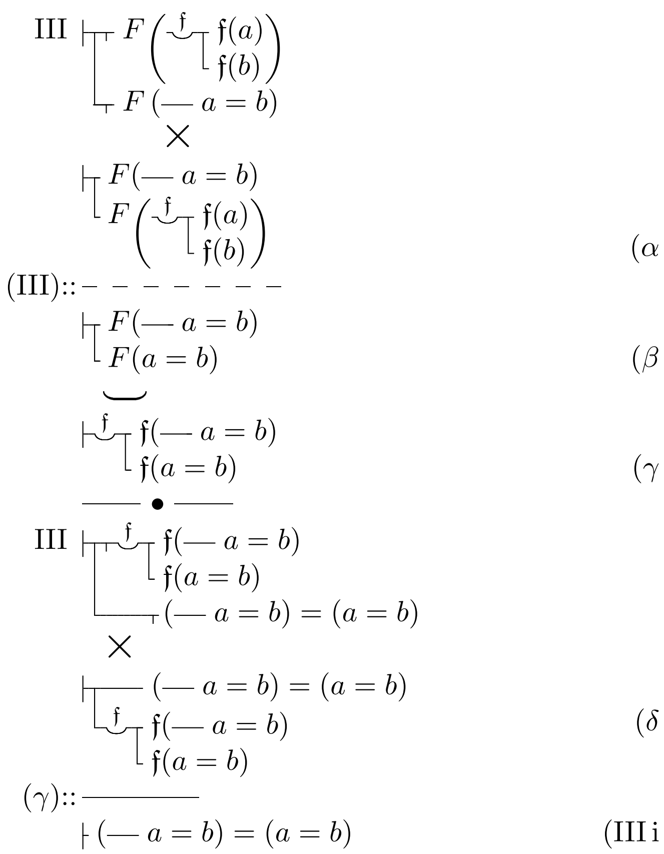
In the following derivation we replace the function-letter 'f' in (III c) by the Roman function-marker '$f(a) = f(\xi)$'.

III c $\vdash$ $f(a) = f(b)$
$f(a) = f(a)$
$a = b$

(III e)::

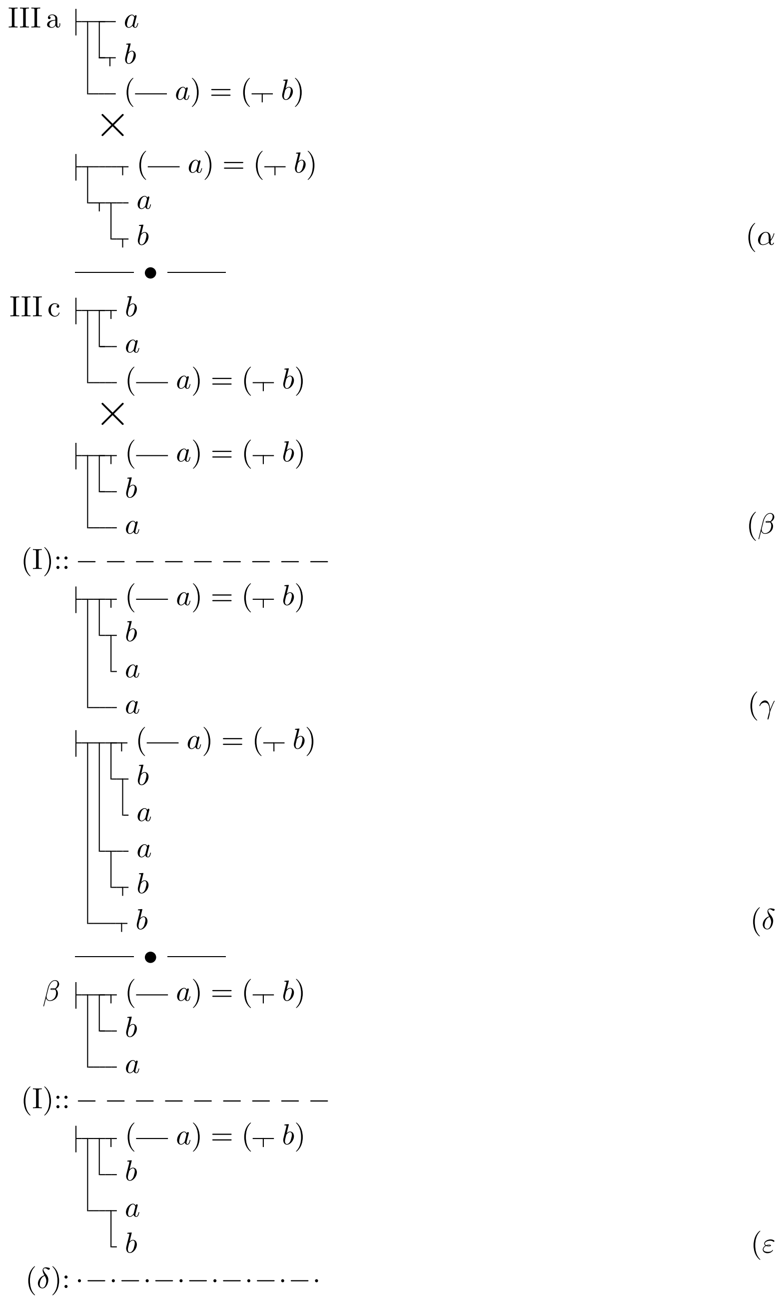
$\vdash$ $f(a) = f(b)$
$a = b$ (III h

In the following derivation the function-letter 'g' in (III) is replaced by '$\top$ F(— ξ)' and the horizontals are fused.

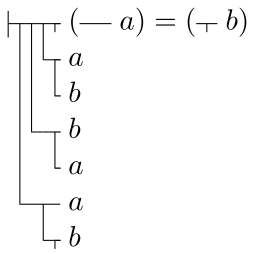


In the second citation of (III), 'g' is replaced by 'F'. In the | final citation of (III), '$g(\xi)$' is replaced by '$\top\ \xi$', 'a' by '— $a = b$', 'b' by '$a = b$'. 68a

§51. Some propositions are now to be derived from (IV).

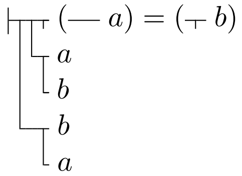


68b



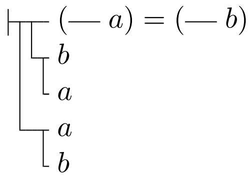
(ζ

(α): · – · – · – · – · – · – ·



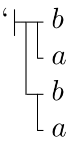
(η

(IV):: – – – – – – – – – – –

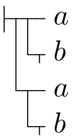


(IV a

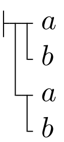
In its first application, (I) is here to be thought of in the form

‘’,

in its second in the form

‘’

and in its third in the form

‘’.

Note the effect of applying (I) in the transitions to (γ), (δ) and (ε). (I) will often be used in this way in what follows. Compare here the derivation of (I e) in §49. Proposition (IV a) will often be used to | prove equality of truth-values. 69a

$$\text{IV} \quad \vdash\!\!\!\begin{array}{l} (\text{—}\, a) = (\top\, a) \\ \llcorner\!\top\, (\text{—}\, a) = (\top\top\, a) \end{array}$$

$$\times$$

$$\vdash\!\!\!\begin{array}{l} (\text{—}\, a) = (\top\top\, a) \\ \llcorner\!\top\, (\text{—}\, a) = (\top\, a) \end{array}$$

(III g)::
____

$$\vdash (\text{—}\, a) = (\top\top\, a) \qquad \text{(IV b}$$

(III a):
____

$$\vdash\!\!\!\begin{array}{l} f(\text{—}\, a) \\ \llcorner\, f(\,\top\top\, a) \end{array} \qquad \text{(IV c}$$

——— • ———

$$\text{IV b} \quad \vdash (\text{—}\, a) = (\top\top\, a)$$

(III c):
____

$$\vdash\!\!\!\begin{array}{l} f(\,\top\top\, a) \\ \llcorner\, f(\text{—}\, a) \end{array} \qquad \text{(IV d}$$

An example of an application of (IV a) is the following:

$$\text{III f} \quad \vdash\!\!\!\begin{array}{l} a = b \\ \llcorner\, b = a \end{array}$$

(IV a):
____

$$\vdash\!\!\!\begin{array}{l} (\text{—}\, a = b) = (\text{—}\, b = a) \\ \llcorner\!\top\, b = a \\ \quad \llcorner\, a = b \end{array} \qquad (\alpha$$

(III f)::
____

$$\vdash (\text{—}\, a = b) = (\text{—}\, b = a)$$

(III c):
____

$$\vdash\!\!\!\begin{array}{l} (a = b) = (\text{—}\, b = a) \\ \llcorner\, (\text{—}\, a = b) = (a = b) \end{array} \qquad (\gamma$$

(III i)::
____

$$\vdash (a = b) = (\text{—}\, b = a) \qquad (\delta$$

(III c):
____

$$\vdash\!\!\!\begin{array}{l} (a = b) = (b = a) \\ \llcorner\, (\text{—}\, b = a) = (b = a) \end{array} \qquad (\varepsilon$$

(III i)::
____

$$\vdash (a = b) = (b = a) \qquad \text{(IV e}$$

In the transition to (γ), (III c) is here to be thought of in the form

$$‘\vdash\!\!\!\begin{array}{l} (a = b) = (\text{—}\, b = a) \\ \llcorner\, (\text{—}\, a = b) = (\text{—}\, b = a) \\ \llcorner\, (\text{—}\, a = b) = (a = b) \end{array}\,’,$$

| where ‘$f(\xi)$’ is replaced by 69b

$$‘\,\xi = (\text{—}\, b = a)\,’,$$

‘a’ by ‘$(\text{—}\, a = b)$’, ‘b’ by ‘$(a = b)$’. For the transition to (ε), we have to think of ‘$f(\xi)$’ in (III c) as replaced by ‘$(a = b) = \xi$’, ‘a’ by ‘$(\text{—}\, b = a)$’, ‘b’ by ‘$(b = a)$’.

§52. Finally, some propositions may be derived from (V) and (VI).

$$\text{V} \quad \vdash (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = g(\mathfrak{a}))$$

(III a): ——————————————

$$\vdash\!\!\top\; \dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)$$
$$\llcorner\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = g(\mathfrak{a})$$

(III h): – – – – – – – – – –

$$\vdash\!\!\top\; F(\dot{\varepsilon}f(\varepsilon)) = F(\dot{\alpha}g(\alpha))$$
$$\llcorner\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = g(\mathfrak{a}) \qquad \text{(V a}$$

——— • ———

$$\text{V} \quad \vdash (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = g(\mathfrak{a}))$$

(III c): ——————————————

$$\vdash\!\!\top\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = g(\mathfrak{a})$$
$$\llcorner\; \dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha) \qquad (\alpha$$

(II a): ——————————

$$\vdash\!\!\top\; f(a) = g(a)$$
$$\llcorner\; \dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha) \qquad \text{(V b}$$

In (II a), ‘$f(\xi)$’ is here to be thought of as replaced by ‘$f(\xi) = g(\xi)$’.

In the following derivation ‘$g(\xi)$’ in (V a) is replaced by ‘$a = \xi$’, while ‘ε’ is written for ‘α’ by rule (11) §48.

$$\text{V a} \quad \vdash\!\!\top\; \dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}(a = \varepsilon)$$
$$\llcorner\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = (a = \mathfrak{a})$$

(III a): – – – – – – – – – –

$$\vdash\!\!\top\!\!\top\; a = \backslash\dot{\varepsilon}f(\varepsilon)$$
$$\llcorner\; a = \backslash\dot{\varepsilon}(a = \varepsilon)$$
$$\llcorner\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = (a = \mathfrak{a}) \qquad (\alpha$$

VI:: ——————————

$$\vdash\!\!\top\; a = \backslash\dot{\varepsilon}f(\varepsilon)$$
$$\llcorner\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = (a = \mathfrak{a}) \qquad \text{(VI a}$$

II. Proofs of the basic laws of cardinal number 70

Preliminaries

| **§53.** Concerning the proofs to follow I would emphasize that the commentaries that I regularly give in advance under the heading 'analysis' are merely intended to serve the convenience of the reader; they could be omitted without compromising the force of the proof, which is to be sought under the heading 'construction' only. 70a

The rules that I cite in the analysis are listed above in §48 under their respective numbers.[a] The laws derived above are collected in a special table | at the end of the volume, together with the basic laws summerised in §47. 70b
In addition, the definitions of section I, 2, and others, are also collected at the end of the volume.

First we prove the proposition:

The cardinal number of a concept is equal to the cardinal number of a second concept, if a relation maps the first into the second, and if the converse of this relation maps the second into the first.

70

A. Proof of the proposition

$$`\vdash\begin{array}{l} \mathrm{ṗ}u = \mathrm{ṗ}v \\ u \frown (v \frown \rangle q) \\ v \frown (u \frown \rangle \not\!\!\rangle q) \end{array}\text{'}$$

a) Proof of the proposition

$$`\vdash\begin{array}{l} w \frown (v \frown \rangle (p \llcorner q)) \\ w \frown (u \frown \rangle p) \\ u \frown (v \frown \rangle q) \end{array}\text{'},$$

§54. *Analysis*

According to definition (Z) the proposition

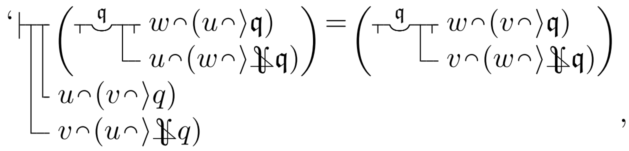
$$`\vdash\begin{array}{l} \mathrm{ṗ}u = \mathrm{ṗ}v \\ u \frown (v \frown \rangle q) \\ v \frown (u \frown \rangle \not\!\!\rangle q) \end{array}\text{'} \qquad (\alpha$$

is a consequence of

$$`\vdash\begin{array}{l} \dot{\varepsilon}\left(\smile^{\mathfrak{q}}\begin{array}{l} \varepsilon \frown (u \frown \rangle \mathfrak{q}) \\ u \frown (\varepsilon \frown \rangle \not\!\!\rangle \mathfrak{q}) \end{array}\right) = \dot{\varepsilon}\left(\smile^{\mathfrak{q}}\begin{array}{l} \varepsilon \frown (v \frown \rangle \mathfrak{q}) \\ v \frown (\varepsilon \frown \rangle \not\!\!\rangle \mathfrak{q}) \end{array}\right) \\ u \frown (v \frown \rangle q) \\ v \frown (u \frown \rangle \not\!\!\rangle q) \end{array}\text{'}, \qquad (\beta$$

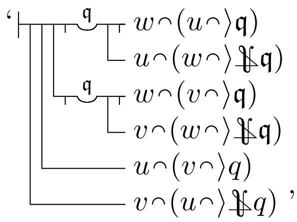
[a] *Translators' Note*: Note that the German is "*Nummern*", not "*Zahlen*".

This proposition is to be derived using (V a) and rule (5) from the proposition
| 71



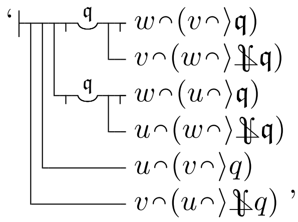
(γ

| which is to be proven using (IV a). For this, we require the propositions 71a



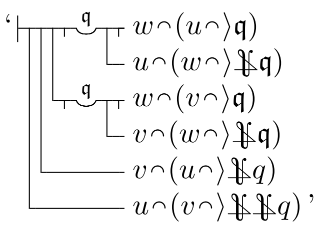
(δ

and



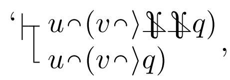
(ε

If we exchange 'u' with 'v' in (ε) and write '⫮q' for 'q', we obtain:



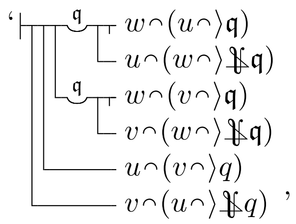
(ζ

This proposition almost coincides with (δ). To derive (δ) from (ζ) by rule (7) we require the proposition



(η

So we first try to prove the proposition (ε). This results by contraposition (rule 3) from the proposition



(ϑ

| which follows by rule (5) from 71b

$$\text{‘}\vdash\begin{array}{l} w\frown(u\frown\rangle p) \\ u\frown(w\frown\rangle\Finv p) \\ \overset{\mathfrak{q}}{\smile}\; w\frown(v\frown\rangle\mathfrak{q}) \\ v\frown(w\frown\rangle\Finv\mathfrak{q}) \\ u\frown(v\frown\rangle q) \\ v\frown(u\frown\rangle\Finv q) \end{array}\text{’} \qquad (\iota$$

In order to grasp the sense of this better, we transform it by contraposition into

$$\text{‘}\vdash\begin{array}{l} \overset{\mathfrak{q}}{\smile}\; w\frown(v\frown\rangle\mathfrak{q}) \\ v\frown(w\frown\rangle\Finv\mathfrak{q}) \\ u\frown(w\frown\rangle\Finv p) \\ w\frown(u\frown\rangle p) \\ u\frown(v\frown\rangle q) \\ v\frown(u\frown\rangle\Finv q) \end{array}\text{’} \qquad (\kappa$$

For ease of expression, I will now say ‘u-concept’ instead of ‘concept whose extension is indicated by ‘u’ ’,[a] ‘p-relation’ instead of ‘relation whose extension is indicated by ‘p’ ’, ‘the p-relation maps the w-concept into the u-concept’ instead of ‘the objects falling under the w-concept are correlated single-valuedly to the objects falling under the u-concept by the p-relation’. We may now rephrase (κ) in words like this:

‘If the converse of the p-relation maps the u-concept into the w-concept and the p-relation maps the w-concept into the u-concept, if further the q-relation maps the u-concept into the v-concept and the $\Finv q$-relation maps the v-concept into the u-concept, then there is | a relation that maps the 72
w-concept into the v-concept and its converse maps the v-concept into the w-concept.’

Such a relation is evidently one that is composed from the p-relation and the q-relation,[1] as the following figure illustrates:

$$w \underset{p}{\twoheadrightarrow} u \underset{q}{\twoheadrightarrow} v$$

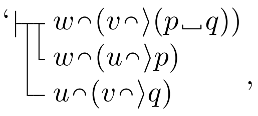
I now introduce the abbreviation ‘$p\llcorner\!\lrcorner q$’ for the extension of a relation which is composed from the p-relation and the q-relation by defining:

$$\Vdash \dot{\alpha}\dot{\varepsilon}\left(\overset{\mathfrak{r}}{\smile}\begin{array}{l} \varepsilon\frown(\mathfrak{r}\frown p) \\ \mathfrak{r}\frown(\alpha\frown q) \end{array}\right) = p\llcorner\!\lrcorner q \qquad (\mathrm{B}$$

It now turns on the proposition

[a] The original actually uses semicircles, ${}_{\text{(}}p^{\text{)}}$, for the quotation-marks *within* the quotation marks, but this is not generally followed in the text. What should we do about this? — also, discuss in introduction

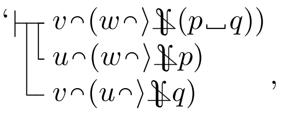
[1] Compare *Grundlagen*, p. 86.

 (λ

in words:

'If the p-relation maps the w-concept into the u-concept and if the q-relation maps the u-concept into the v-concept, then the $(p \smile q)$-relation that is composed from the two maps the w-concept into the v-concept.'

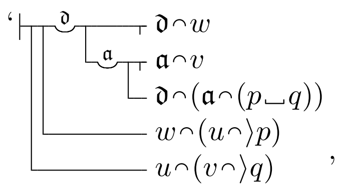
In addition, we require the proposition

 (μ

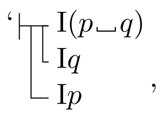
that can be traced back to (λ) using the proposition

$$\text{‘} \vdash \breve{}(p \smile q) = \breve{}q \smile \breve{}p \text{’} \qquad (\nu$$

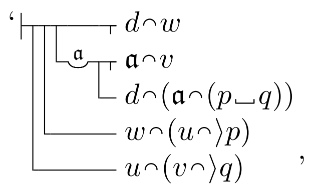
We first attempt to prove the proposition (λ). | From definition (Δ) it can be gathered that two things must be proven, namely first 72b

 (ξ

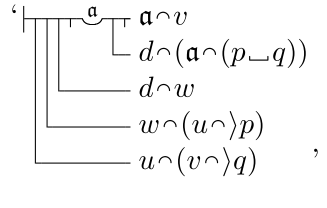
and second

 (o

(ξ) results from

 (π

according to rule (5). In order to express (π) in words, it is convenient first to transform it by contraposition into

 (ϱ

Letting ‘d’ now abbreviate ‘object which is indicated by ‘d’ ’,[b] our proposition may be rendered in words like this:

‘If d falls under the w-concept, and if the w-concept is mapped into the u-concept by the p-relation, and if the u-concept is mapped into the v-concept by the q-relation, then there is an object that falls under the v-concept and that stands in the $(p \smile q)$-relation to d.’

| The proof will need to rely on the proposition 73a

$$‘\vdash \begin{array}{l} d \frown (m \frown (p \smile q)) \\ \llcorner\, e \frown (m \frown q) \\ \llcorner\, d \frown (e \frown p) \end{array}’ \qquad (\sigma$$

In words:

‘If d stands in the p-relation to e, and if e stands in the q-relation to m, then d stands in the $(p \smile q)$-relation to m.’

This will need to be derived from the proposition[a]

$$‘\vdash \left(\stackrel{\mathfrak{r}}{\smile} \begin{array}{l} d \frown (\mathfrak{r} \frown p) \\ \llcorner\, \mathfrak{r} \frown (m \frown q) \end{array} \right) = d \frown (m \frown (p \smile q))’ \qquad (\tau$$

which follows from definition (B). In order to prove it, we require the proposition

$$‘\vdash f(a, b) = a \frown (b \frown \dot{\alpha}\dot{\varepsilon} f(\varepsilon, \alpha))’ \qquad (\upsilon$$

since the left-hand side of the definitional equation (B) is a double value-range. (υ) is to be traced back to the proposition

$$‘\vdash f(a) = a \frown \dot{\varepsilon} f(\varepsilon)’ \qquad (\varphi$$

which is to be derived from definition (A). Accordingly, we have to prove

$$‘\vdash f(a) = \backslash \dot{\alpha} \left(\stackrel{\mathfrak{g}}{\smile} \begin{array}{l} \mathfrak{g}(a) = \alpha \\ \llcorner\, \dot{\varepsilon} f(\varepsilon) = \dot{\varepsilon}\mathfrak{g}(\varepsilon) \end{array} \right)’ \qquad (\chi$$

This has to be done by appeal to (VI a) and the proposition[b]

$$‘\vdash\!\stackrel{\mathfrak{a}}{\smile} \left(\stackrel{\mathfrak{g}}{\smile} \begin{array}{l} \mathfrak{g}(a) = \mathfrak{a} \\ \llcorner\, \dot{\varepsilon} f(\varepsilon) = \dot{\varepsilon}\mathfrak{g}(\varepsilon) \end{array} \right) = (f(a) = \mathfrak{a})’ \qquad (\psi$$

taking

$$‘\stackrel{\mathfrak{g}}{\smile} \begin{array}{l} \mathfrak{g}(a) = \xi \\ \llcorner\, \dot{\varepsilon} f(\varepsilon) = \dot{\varepsilon}\mathfrak{g}(\varepsilon) \end{array}’$$

---

[b]Regarding the quotation marks within the quotation, see footnote a on p. 71 above.

[a] *Translators' Note*: Olms-reprint (but not the original) misses out the opening quotation mark and the judgement-stroke of (τ).

[b] *Translators' Note*: Olms-reprint (but not the original) misses out the opening quotation mark and the judgement-stroke of (ψ).

for '$f(\xi)$' in (VI a) and replacing 'a' by '$f(a)$'. (ψ) is obtained by rule (5) from

$$\text{`}\vdash\left(\;\overset{\mathfrak{g}}{\smile}\begin{array}{l}\mathfrak{g}(a) = b\\ \dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\right) = (f(a) = b)\text{'} \qquad (\omega$$

| which is to be proven by appeal to (IV a). For this, we then require the propositions 73b

$$\text{`}\vdash\begin{array}{l}\overset{\mathfrak{g}}{\smile}\begin{array}{l}\mathfrak{g}(a) = b\\ \dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\\ f(a) = b\end{array}\text{'} \qquad (\alpha'$$

and

$$\text{`}\vdash\begin{array}{l}f(a) = b\\ \overset{\mathfrak{g}}{\smile}\begin{array}{l}\mathfrak{g}(a) = b\\ \dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\end{array}\text{'} \qquad (\beta'$$

the first of which follows by contraposition according to rule (3) from

$$\text{`}\vdash\begin{array}{l}\top f(a) = b\\ \overset{\mathfrak{g}}{\smile}\begin{array}{l}\mathfrak{g}(a) = b\\ \dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\end{array}\text{'} \qquad (\gamma'$$

If we now write (II b) in the form

$$\text{`}\vdash\begin{array}{l}\begin{array}{l}f(a) = b\\ \dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}f(\varepsilon)\end{array}\\ \overset{\mathfrak{g}}{\smile}\begin{array}{l}\mathfrak{g}(a) = b\\ \dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\end{array}\text{'}$$

then we see that (γ') follows from it and (III e). The proposition (β') follows by contraposition from

$$\text{`}\vdash\begin{array}{l}\overset{\mathfrak{g}}{\smile}\begin{array}{l}\mathfrak{g}(a) = b\\ \dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}\mathfrak{g}(\varepsilon)\end{array}\\ \top f(a) = b\end{array}\text{'} \qquad (\delta'$$

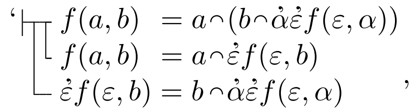
and this in turn by rule (5) from

$$\text{`}\vdash\begin{array}{l}\begin{array}{l}g(a) = b\\ \dot{\varepsilon}f(\varepsilon) = \dot{\varepsilon}g(\varepsilon)\end{array}\\ \top f(a) = b\end{array}\text{'} \qquad (\varepsilon'$$

This proposition is obtained by rule (7) and (V b) from

$$\text{`}\vdash\begin{array}{l}\begin{array}{l}g(a) = b\\ f(a) = g(a)\end{array}\\ \top f(a) = b\end{array}\text{'}$$

which, by permutation of subcomponents, is just a special case of (III c). We now construct the proof accordingly. Concerning the derivation of (2), it is to be remarked in addition that | in the first citation of (1) by rule 74a
(9), ‘$f(\xi)$’ is replaced by the function-marker ‘$f(\xi, b)$’. Next, (III c) is to be thought of in the form



In the second citation of (1), it is to be thought of in the form

$$‘\vdash \dot{\varepsilon} f(\varepsilon, b) = b \frown \dot{\alpha}\dot{\varepsilon} f(\varepsilon, \alpha)’$$

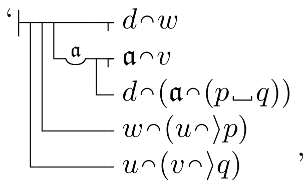
by putting ‘$\dot{\varepsilon} f(\varepsilon, \xi)$’ for ‘$f(\xi)$’, ‘$b$’ for ‘$a$’, and ‘$\alpha$’ for ‘$\varepsilon$’, according to the rules (9) and (11).

| **§55.** *Construction* 74b

[proof to be added]

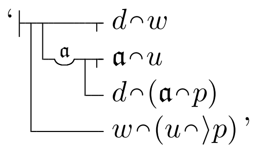
§56. *Analysis* 76a

Now, in order to prove the proposition



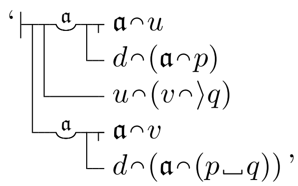
(α

(§54, π) we have to go back to (Δ). From this we derive the proposition



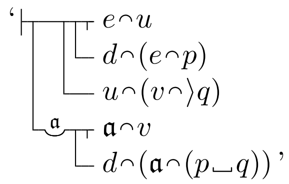
(β

In order to reach (α) from this, we need to have the proposition



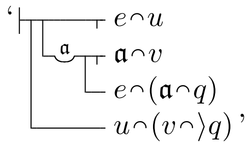
(γ

which we obtain by rule (5) from



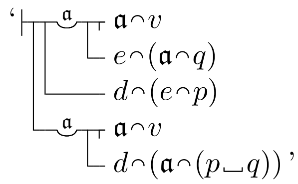
(δ

We now can also write the proposition (β) like this:



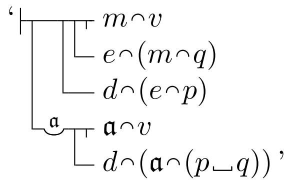
(β

To attain (δ) from this, we require the proposition



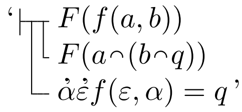
(ε

| which follows by rule (5) from 76b



(ζ

This proposition is easily proven by means of (II a) and (5). It thus turns on deriving the proposition (β) from (Δ). This is accomplished by means of



(η

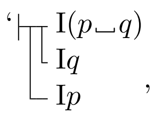
which follows from (3).

§57. *Construction*

[proof to be added]

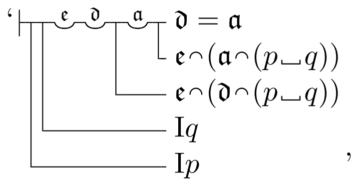
§58. *Analysis* 77b

We now have to prove the proposition (§54, o)



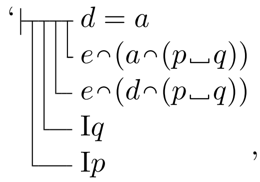
(α

i.e.: 'the relation that is composed from the p-relation and the q-relation is single-valued if both the p-relation | and the q-relation are single-valued.' 78a
By definition (Γ), what has to be proven is



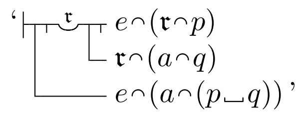
(β

which is obtained by rule (5) from



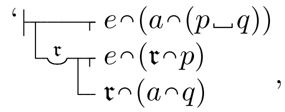
(γ

From definition (B) it is easy to deduce



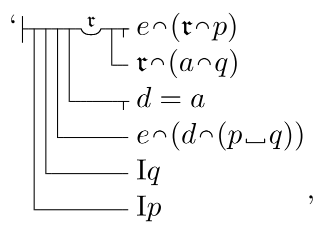
(δ

or



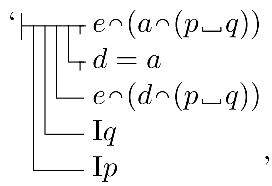
(ε

With the latter, one easily passes from



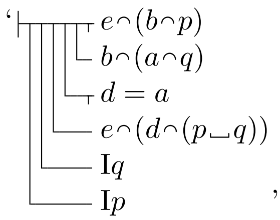
(ζ

to the proposition



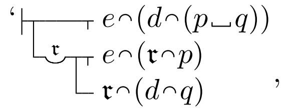
(η

from which (γ) follows by contraposition. The proposition (ζ) is obtained by rule (5) from | 78b



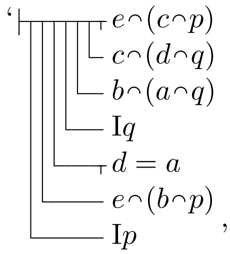
$(\vartheta$

This proposition is to be derived by means of



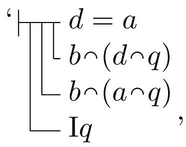
$(\varepsilon$

from



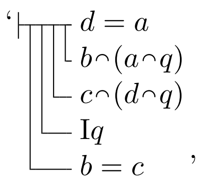
$(\iota$

in a manner similar to how (η) is derived from (ϑ). Here 'c' will need to be replaced by '𝔯'. Therefore the letters 'b' and 'c' must be distinct; otherwise, by rule (5), '𝔯' would be introduced not only at places now occupied by 'c', but also at those occupied by 'b'. Now, from our definition (Γ)



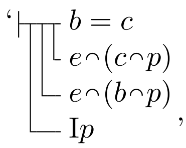
$(\kappa$

follows and from this by means of (III c)



$(\lambda$

If the proposition (κ) is applied to this in the form



$(\kappa$

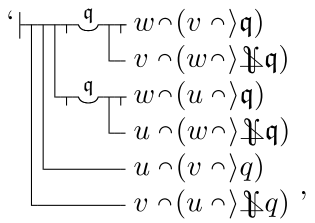
| then the proposition (ι) is proven, on the basis of which we can reach our proposition (α), as we saw. 79a

§59. *Construction*

[proof to be added]

80

b) Proof of the proposition



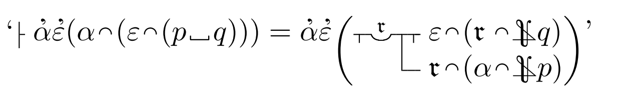
and end of section A

| **§60.** *Analysis* 81

We now have to prove the proposition (§54, ν)

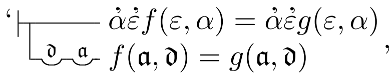
$$\text{‘}\vdash \nparallel(p \smile q) = \nparallel q \smile \nparallel p\text{’} \qquad (\alpha$$

According to definitions (E) and (B), this comes down to deriving the proposition



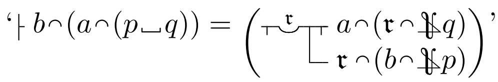
(β

For this we may draw upon the proposition



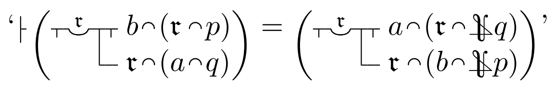
(γ

which we will also use on other occasions and which can be proven by a double application of the proposition (V a). In order to apply this proposition here, we have to obtain the proposition



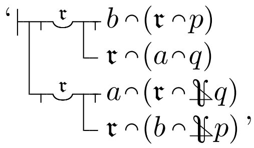
(δ

which follows by (4) from



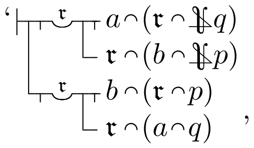
(ε

| (ε) is to be proven by means of (IV a). For this we need the propositions 81a



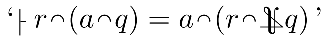
(ζ

and



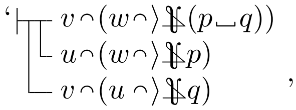
(η

We derive them from the proposition



(ϑ

that straightforwardly follows from (E). The proposition (α), thus proven, will be deployed as indicated in §54 in the proof of the proposition (§54, μ)



(ι

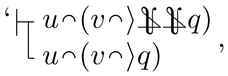
and from this and (19) we derive the proposition (§54, ε).

§61. *Construction*

[proof to be added]

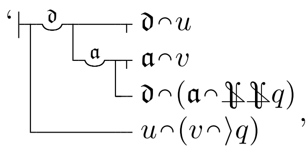
§62. *Analysis*

In order to derive the proposition (§54, δ) from (25), the proposition
(§54, η) is needed | 84a



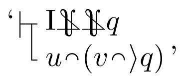
(α

For this, according to (11), we have to prove the propositions



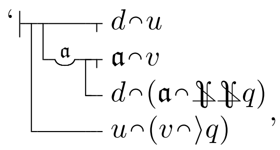
(β

and



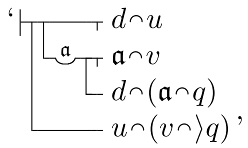
(γ

(β) is obtained by rule (5) from



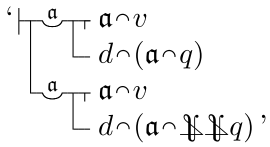
(δ

By (8) we now have



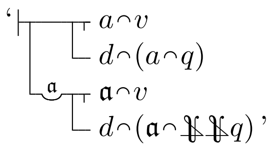
(ε

What thus remains to be proven is



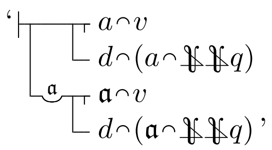
(ζ

which, by rule (5), follows from



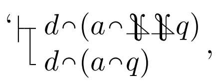
(η

If we write (II a) in this way



(ϑ

then we can see that the proposition



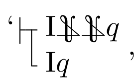
(ι

is to be derived, which is easily done by (22).

| **§63.** *Construction* 84b

[proof to be added]

§64. *Analysis*
We are still lacking the proposition (§62, γ). We first prove



(α

from which the former follows together with (18). According to (16) we have to derive

$$\text{`}\vdash\overset{\mathfrak{e}}{\smile}\overset{\mathfrak{d}}{\smile}\overset{\mathfrak{a}}{\smile}\begin{array}{l}\mathfrak{d}=\mathfrak{a}\\ \mathfrak{e}\frown(\mathfrak{a}\frown\rangle\nmid\!\nmid q)\\ \mathfrak{e}\frown(\mathfrak{d}\frown\rangle\nmid\!\nmid q)\\ \mathrm{I}q\end{array}\text{'} \qquad (\beta$$

| or 85a

$$\text{`}\vdash\begin{array}{l} d=a\\ e\frown(a\frown\nmid\!\nmid q)\\ e\frown(d\frown\nmid\!\nmid q)\\ \mathrm{I}q\end{array}\text{'} \qquad (\gamma$$

By (13) we now have

$$\text{`}\vdash\begin{array}{l} d=a\\ e\frown(a\frown q)\\ e\frown(d\frown q)\\ \mathrm{I}q\end{array}\text{'} \qquad (\delta$$

(γ) follows from this and the proposition

$$\text{`}\vdash\begin{array}{l} e\frown(a\frown q)\\ e\frown(a\frown\nmid\!\nmid q)\end{array}\text{'} \qquad (\varepsilon$$

which follows from (23) in a manner similar to that in which (26) follows from (22). After we have thus proven proposition (§62, γ), we will employ it to eliminate the subcomponent '$\mathrm{I}\nmid\!\nmid q$' in (27) by fusion of subcomponents. Thus we arrive at the goal of our section A, as stated in §54.

§65. *Construction*

[proof to be added]

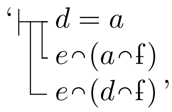
| **§66.** *Analysis* 87a

In order to prove the proposition that the relation of a cardinal number to the one immediately following is single-valued, or as one can also say, that for every cardinal number there is no more than one which immediately follows it in the cardinal number series,[1] we have to use proposition (16) and so we have to derive

$$\text{`}\vdash\overset{\mathfrak{e}}{\smile}\overset{\mathfrak{d}}{\smile}\overset{\mathfrak{a}}{\smile}\begin{array}{l}\mathfrak{d}=\mathfrak{a}\\ \mathfrak{e}\frown(\mathfrak{a}\frown\mathfrak{f})\\ \mathfrak{e}\frown(\mathfrak{d}\frown\mathfrak{f})\end{array}\text{'} \qquad (\alpha$$

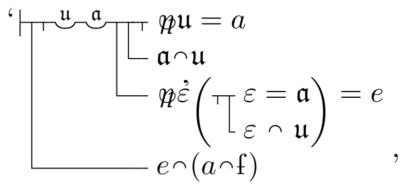
which follows from

[1]Compare §43.



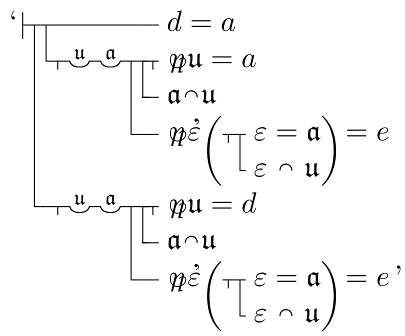
(β

From definition (*H*) the following is easily derived



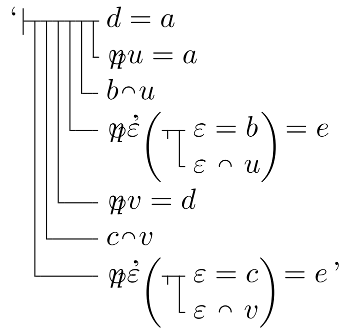
(γ

Accordingly, what needs to be proven is



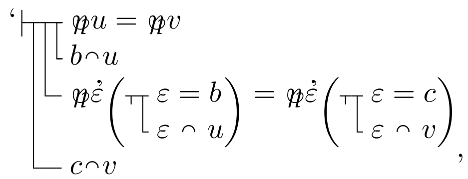
(δ

a proposition which, by multiple applications of contraposition and the introduction of German letters, is obtained from | 87b



(ε

The latter proposition can be derived from



(ζ

According to the recently proven proposition (32), we now merely need to show that there is a relation which maps the *u*-concept into the *v*-concept and whose converse maps the *v*-concept into the *u*-concept. That there

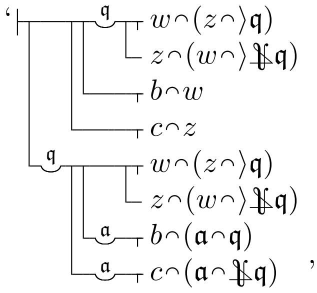
is a relation which maps the $\dot{\varepsilon}\left(\begin{array}{l}\top\!\!\!\top\, \varepsilon = b \\ \;\llcorner\, \varepsilon \frown u\end{array}\right)$-concept into the $\dot{\varepsilon}\left(\begin{array}{l}\top\!\!\!\top\, \varepsilon = c \\ \;\llcorner\, \varepsilon \frown v\end{array}\right)$-concept follows from the equality of the cardinal numbers belonging to these concepts, which of course still has to be proven. Now, the v-concept differs in its extension from the $\dot{\varepsilon}\left(\begin{array}{l}\top\!\!\!\top\, \varepsilon = c \\ \;\llcorner\, \varepsilon \frown v\end{array}\right)$-concept only in that the object c falls under it but not the latter; and the u-concept differs in its extension from the $\dot{\varepsilon}\left(\begin{array}{l}\top\!\!\!\top\, \varepsilon = b \\ \;\llcorner\, \varepsilon \frown u\end{array}\right)$-concept only in that the object b falls under it but not the latter. From this it must now be possible to conclude | that there is also a 88a
relation that maps the u-concept into the v-concept. If one were to follow the usual practice of mathematicians, one might say something like this: we correlate the objects, other than b, falling under the u-concept with the objects, other than c, falling under the v-concept by means of the known relation, and we correlate b with c. In this way, we have mapped the u-concept into the v-concept and, conversely, the latter into the former. So, according to the proposition just proven, the cardinal numbers that belong to them are equal. This is indeed much briefer than the proof to follow which some, misunderstanding my project, will deplore on account of its length. What is it that we are doing when we correlate objects for the purpose of a proof? Seemingly something similar to drawing an auxiliary line in geometry. Euclid, whose method can still often serve as a model of rigour, has his postulates for this purpose, stating that certain lines may be drawn. However, the drawing of a line should no more be regraded as a creation, than the specification of a point of intersection. Rather, in both cases we merely bring to attention, apprehend, what is already there. What is essential to a proof is only that there be such a thing. In proofs, Euclid's postulates thus have the force of axioms that assert that there are certain lines, certain points. Since here we are aiming to reach down to the
| deepest foundations in every case, we ask on what the possibility of such 88b
correlation is based. If one wanted to propose a postulate, in the style of Euclid, it might be phrased like this: 'It is postulated that any object is correlated with any object' or 'It is possible to correlate any object with any object'. This should no more be regarded as a psychological proposition, than should Euclid's postulates be regarded as asserting an ability of our minds; for, so understood, it would indeed be false, since not all objects are known to us and the same ones are not known to everyone. In this way, a subjective element would intrude, which is completely alien to the subject-matter. Correlations also have to be possible of infinitely many with infinitely many objects, but only a few of these infinitely many correlations could actually be carried out if correlating were a creative activity of the mind. Rather, the postulates would have to be understood in this way: 'Any object is correlated with any object' or 'There is a correlation between any object and any object'. What then is such a correlation if it is nothing

subjective, created only by our making? However, a particular correlation of an object to an object is not what can be at issue here, and what corresponds to an auxiliary line in geometry; rather we require a genus of correlations, so to speak, that which | we have so-far called relations and will continue to 89a
do so. The desired correlation is thus achieved if we have found[1] a relation between an object b and an object c which also maps the $\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = b \\ \;\llcorner\, \varepsilon \frown u\end{array}\right)$-concept into the $\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = c \\ \;\llcorner\, \varepsilon \frown v\end{array}\right)$-concept and whose converse maps the latter into the former. Here a q-relation which maps the $\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = b \\ \;\llcorner\, \varepsilon \frown u\end{array}\right)$-concept into the $\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = c \\ \;\llcorner\, \varepsilon \frown v\end{array}\right)$-concept, and whose converse maps the latter into the former, may be presupposed as known. What is not known, however, is whether b stands in this relation to any object, nor whether any object stands in this relation to c. We can now give a relation in which every pair of q-related objects stand, and by which b is related to c. This is the $\dot{\alpha}\dot{\varepsilon}\left(\begin{array}{l}\top\!\!-\!\!-\, \varepsilon \frown (\alpha \frown q) \\ \llcorner\!\top\, \varepsilon = b \\ \;\llcorner\, \alpha = c\end{array}\right)$-relation.[2] Although this | has the other desired 89b
properties, it cannot be said whether it and its converse are single-valued as long as nothing more specific is known of the q-relation. For example, it might be that b stands in the q-relation to an object d distinct from c. Then, b would stand in the given relation to two objects, namely c and d, and so would not be single-valued, even though the q-relation is single-valued by assumption. In order to avoid this, we will seek a relation which shares the properties of the q-relation that are desirable for our purposes, but in which b stands to no object and in which no object stands to c. The $\dot{\alpha}\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\!\!-\, \varepsilon \frown (\alpha \frown q) \\ \llcorner\!\top\, \varepsilon = b \\ \;\llcorner\!\top\, \alpha = c\end{array}\right)$-relation is such a relation. If we first abbreviate '$\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = b \\ \;\llcorner\, \varepsilon \frown u\end{array}\right)$' by '$w$' and '$\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = c \\ \;\llcorner\, \varepsilon \frown v\end{array}\right)$' by '$z$', what we have to prove is the proposition: 'If there is a q-relation which maps the w-concept into the z-concept and whose converse maps the latter into the former, then there is also a relation which does the same, but in which b stands to no object and in which no object stands to c, provided b does not fall under the w-concept and c does not fall under the z-concept.'

[1]To pursue the model of geometry, one could say 'constructed'; however, one has to remember that this is not creation.

[2]Without essential changes, we may write '$\begin{array}{l}\top\!\!-\!\!-\, x \frown (y \frown q) \\ \llcorner\!\top\!\top\!\top\, x = b \\ \;\llcorner\, y = c\end{array}$' instead of '$\begin{array}{l}\top\!\!-\!\!-\, x \frown (y \frown q) \\ \llcorner\!\top\, x = b \\ \;\llcorner\, y = c\end{array}$'. Compare here what is said about 'or' and 'and' in §12.

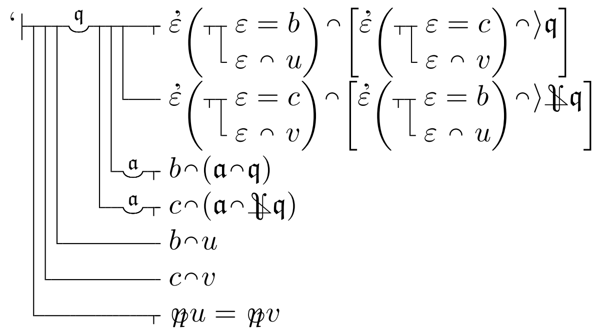
For the derivation in concept-script it is more convenient to prove the proposition which results from the latter by contraposition: | 90a



(η

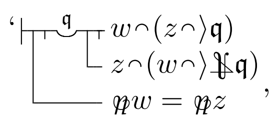
Next, we will derive the proposition: 'If there is a relation which maps the $\dot{\varepsilon}\left(\begin{smallmatrix}\top\ \varepsilon = b\\ \varepsilon \frown u\end{smallmatrix}\right)$-concept into the $\dot{\varepsilon}\left(\begin{smallmatrix}\top\ \varepsilon = c\\ \varepsilon \frown v\end{smallmatrix}\right)$-concept | and whose converse 90b
maps the latter into the former, and which is so constituted that b stands in it to no object and that no object stands in it to c, then there is a relation which maps the u-concept into the v-concept and whose converse maps the latter into the former, provided b falls under the u-concept and c falls under the v-concept.'

The consequent can also be expressed as: 'then the cardinal number of the u-concept is equal to the cardinal number of the v-concept'. Contraposing, the proposition looks like this:



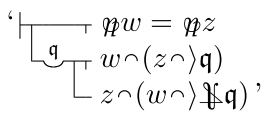
(ϑ

| From the propositions (η) and (ϑ) together with the above proposition 90a



(ι

or



(κ

our proposition (ζ) follows. The proposition (κ) follows straightforwardly from (Z) and

$$\text{‘}\vdash\!\!\begin{array}{l} w\frown\mathfrak{p}z \\ \mathfrak{p}w = \mathfrak{p}z \end{array}\text{’} \qquad\qquad (\lambda$$

and the latter from (III c) in the form

$$\text{‘}\vdash\!\!\begin{array}{l} w\frown\mathfrak{p}z \\ w\frown\mathfrak{p}w \\ \mathfrak{p}w = \mathfrak{p}z \end{array}\text{’}$$

and the proposition

$$\text{‘}\vdash w\frown\mathfrak{p}w\text{’} \qquad\qquad (\mu$$

| This proposition is straightforwardly proven by showing that equality is a relation that maps every concept into itself and whose converse does the same. Accordingly, these propositions are to be derived: 90b

$$\text{‘}\vdash w\frown(w\frown\rangle\dot{\alpha}\dot{\varepsilon}(\varepsilon = \alpha))\text{’} \qquad\qquad (\nu$$

$$\text{‘}\vdash w\frown(w\frown\rangle\mathfrak{U}\dot{\alpha}\dot{\varepsilon}(\varepsilon = \alpha))\text{’} \qquad\qquad (\xi$$

Instead of (ν), we first prove the somewhat more comprehensive proposition,

$$\text{‘}\vdash\!\!\begin{array}{l} u\frown(v\frown\rangle\dot{\alpha}\dot{\varepsilon}(\varepsilon = \alpha)) \\ \smile^{\mathfrak{a}}\ (\text{—}\ \mathfrak{a}\frown u) = (\text{—}\ \mathfrak{a}\frown v) \end{array}\text{’} \qquad\qquad (o$$

which we will also use later. For this we need the propositions

$$\text{‘}\vdash\!\!\begin{array}{l} d\frown u \\ \smile^{\mathfrak{a}}\!\!\begin{array}{l} \mathfrak{a}\frown v \\ d\frown(\mathfrak{a}\frown\dot{\alpha}\dot{\varepsilon}(\varepsilon = \alpha) \end{array} \\ \smile^{\mathfrak{a}}\ (\text{—}\ \mathfrak{a}\frown u) = (\text{—}\ \mathfrak{a}\frown v) \end{array} \qquad\qquad (\pi$$

| and 91a

$$\text{‘}\vdash \mathrm{I}\dot{\alpha}\dot{\varepsilon}(\varepsilon{=}\alpha)\text{’} \qquad\qquad (\varrho$$

The latter follows from (III c) and (16) using (2). In order to prove (π), we write (II a) thus:

$$\text{‘}\vdash\!\!\begin{array}{l} d\frown v \\ d\frown(d\frown\dot{\alpha}\dot{\varepsilon}(\varepsilon = \alpha)) \\ \smile^{\mathfrak{a}}\!\!\begin{array}{l} \mathfrak{a}\frown v \\ d\frown(\mathfrak{a}\frown\dot{\alpha}\dot{\varepsilon}(\varepsilon = \alpha)) \end{array} \end{array}\text{’}$$

It remains to be shown that

$$\text{‘}\vdash d\frown(d\frown\dot{\alpha}\dot{\varepsilon}(\varepsilon = \alpha))\text{’} \qquad\qquad (\sigma$$

which can be inferred from (III e) and (2). The supercomponent ‘$\top\, d\frown v$’ may, by means of the subcomponent

$$‘\ \overset{\mathfrak{a}}{\smile}(—\,\mathfrak{a}\frown u) = (—\,\mathfrak{a}\frown v)\ ’$$

easily be transformed into ‘$\top\, d\frown u$’.

§67. *Construction*

[proof to be added]

92a

§68. *Analysis*

In order to prove the proposition

$$‘\vdash w\frown(w\frown\rangle\!\!\!\setminus\dot{\alpha}\dot{\varepsilon}(\varepsilon=\alpha))\ ’$$

we make use of the proposition

$$‘\vdash \dot{\alpha}\dot{\varepsilon}(\varepsilon=\alpha) = \rangle\!\!\!\setminus\dot{\alpha}\dot{\varepsilon}(\varepsilon=\alpha)\ ’$$

which we can derive from the propositions

$$‘\vdash \dot{\alpha}\dot{\varepsilon}f(\alpha,\varepsilon) = \rangle\!\!\!\setminus\dot{\alpha}\dot{\varepsilon}f(\varepsilon,\alpha)\ ’$$
$$‘\vdash \dot{\alpha}\dot{\varepsilon}(\varepsilon=\alpha) = \dot{\alpha}\dot{\varepsilon}(\alpha=\varepsilon)\ ’$$

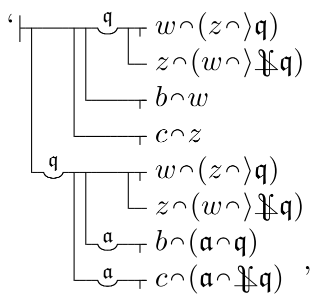
| The former may be derived from (2) and definition (E), the latter from (IVe), in both cases by use of (20). 92b

§69. *Construction*

[proof to be added]

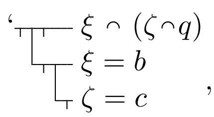
§70. *Analysis*

We now prove the proposition

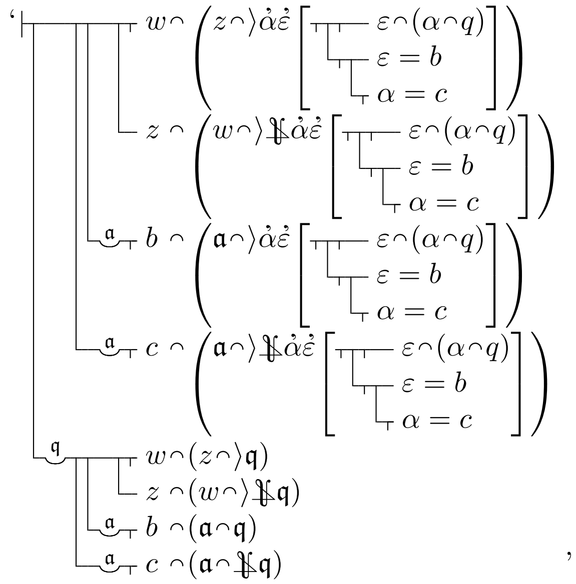


(Compare §66, η.)

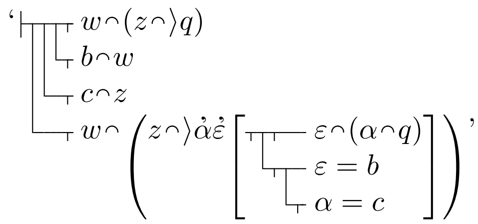
We saw in §66 that



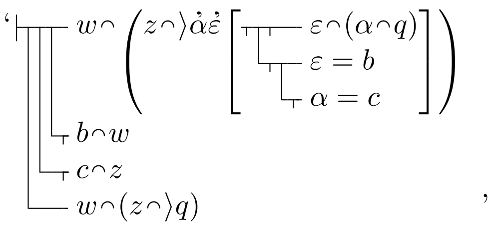
indicates a relation that agrees with the q-relation in the relevant properties, but in which b stands to no object, and in which no object stands to c. We thus write (II a) in the form 94



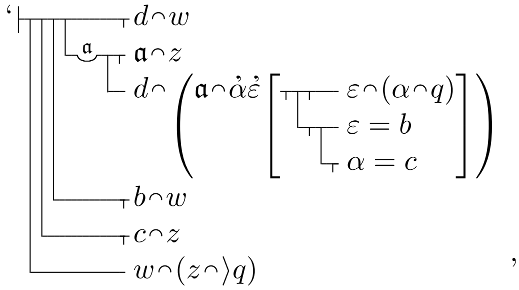
| and now have to prove, amongst others, the proposition: 94a



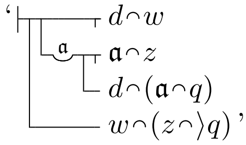
which follows by contraposition from



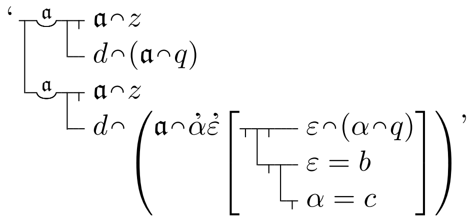
For this, we require the proposition



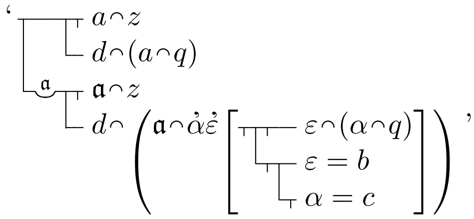
| By (8) we now have 94b



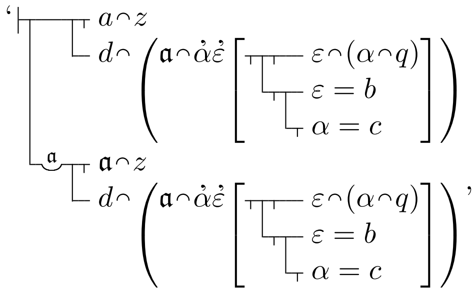
Accordingly, what needs to be proven is something like



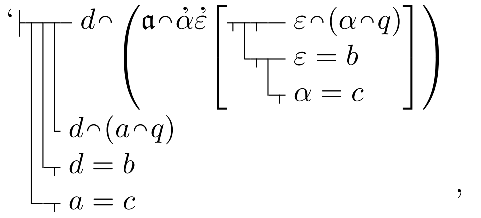
in which I have not yet put a judgement-stroke, because of conditions (subcomponents) that may have to be added. In the proper proof, expressions containing Roman letters may not occur without a judgement-stroke; here, where we concerned with a preliminary exploration, they may be admissible. The latter can be obtained by rule (5) from an expression like | 95a



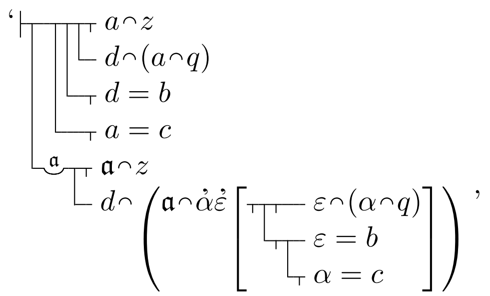
According to (II a), we now have



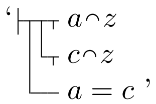
and it would remain to prove, using (36),



in which the subcomponents '⊤ $d = b$' and '⊤ $a = c$' occur. From the latter two propositions we infer by rule (7)



If we now attempted to introduce the German '$\mathfrak{a}$' instead of the Roman 'a' here, by rule (5), we would not attain the desired goal because the subcomponent, '⊤ $a = c$', would have to be included in the scope of '$\mathfrak{a}$'. | But now, 95b
using (III a), we have



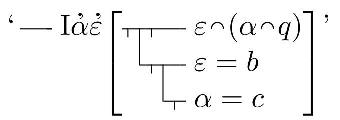
and by rule (8), the subcomponent '⊤ $a = c$' can be replaced by '⊤ $c \frown z$'. Later, the subcomponent '⊤ $d = b$' is likewise to be replaced by '⊤ $b \frown w$'.

§71. *Construction*

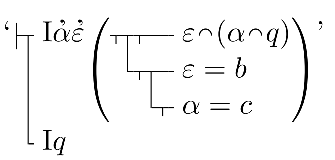
[proof to be added]

§72. *Analysis* 96a

In the proposition (50), we have to remove the subcomponent



This is achieved by using the proposition 96b



for whose proof we use (34). For this we require the proposition 97a

$$‘\vdash \begin{array}{l} d = a \\ e \frown (a \frown q) \\ e = b \\ a = c \\ e \frown (d \frown q) \\ e = b \\ d = c \\ \mathrm{I}q \end{array} ’,$$

which straightforwardly follows from (13).

§73. *Construction*

[proof to be added]

§74. *Analysis* 97b

In order now to prove in addition the proposition

$$‘\vdash \begin{array}{l} z \frown \left(w \frown \rangle \mathfrak{U} \dot{\alpha}\dot{\varepsilon} \left[\begin{array}{l} \varepsilon \frown (\alpha \frown q) \\ \varepsilon = b \\ \alpha = c \end{array} \right] \right) \\ c \frown z \\ b \frown w \\ z \frown (w \frown \rangle \mathfrak{U} q) \end{array} ’,$$

we first write (51) thus:

$$‘\vdash \begin{array}{l} z \frown \left(w \frown \rangle \dot{\alpha}\dot{\varepsilon} \left[\begin{array}{l} \varepsilon \frown (\alpha \frown \mathfrak{U} q) \\ \varepsilon = c \\ \alpha = b \end{array} \right] \right) \\ c \frown z \\ b \frown w \\ z \frown (w \frown \rangle \mathfrak{U} q) \end{array} ’,$$

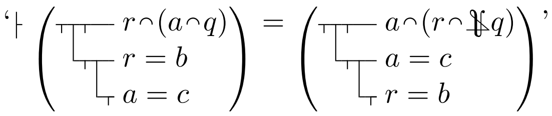
exchanging ‘q’ with ‘$\mathfrak{U} q$’, ‘c’ with ‘b’, ‘z’ with ‘w’. We now have to prove

$$‘\vdash \mathfrak{U} \dot{\alpha}\dot{\varepsilon} \left(\begin{array}{l} \varepsilon \frown (\alpha \frown q) \\ \varepsilon = b \\ \alpha = c \end{array} \right) = \dot{\alpha}\dot{\varepsilon} \left(\begin{array}{l} \varepsilon \frown (\alpha \frown \mathfrak{U} q) \\ \varepsilon = c \\ \alpha = b \end{array} \right)’$$

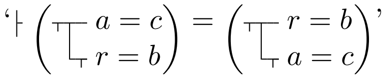
| which follows from 98

$$‘\vdash \dot{\alpha}\dot{\varepsilon} \left(\begin{array}{l} \alpha \frown (\varepsilon \frown q) \\ \alpha = b \\ \varepsilon = c \end{array} \right) = \dot{\alpha}\dot{\varepsilon} \left(\begin{array}{l} \varepsilon \frown (\alpha \frown \mathfrak{U} q) \\ \varepsilon = c \\ \alpha = b \end{array} \right)’$$

using (40). This proposition is to be proven using (20). For this, we require the proposition



which follows from (21) and the proposition



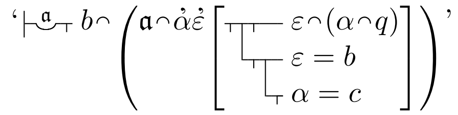
This is to be proven using (IV a).

§75. *Construction*

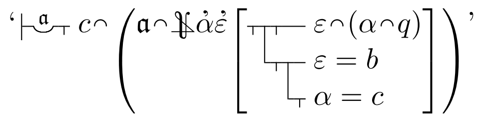
[proof to be added]

| **§76.** *Analysis* 100a

If we write the proposition (II a) as in §70, then we see that the propositions



and

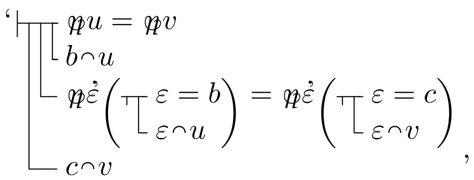


are still missing, of which the latter can be traced back using (53) to the former, which in turn can be derived using (33).

§77. *Construction*

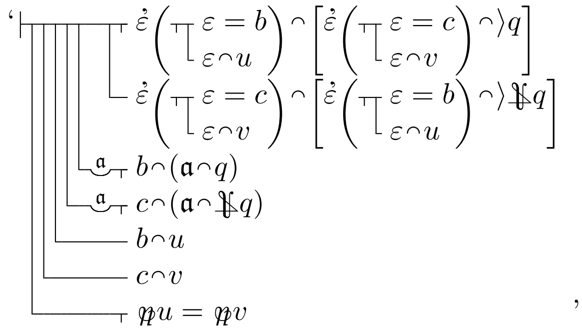
[proof to be added]

b) Proof of the proposition



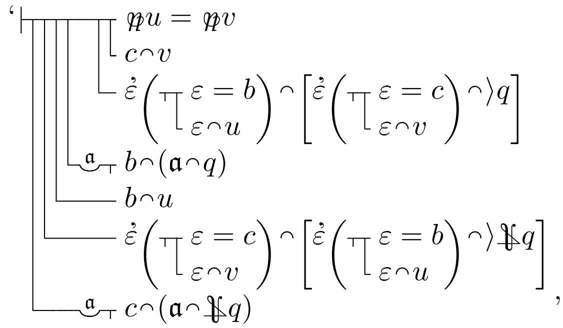
and end of section B

We now prove the proposition (ϑ) of §66 which, by rule (5), is obtained from



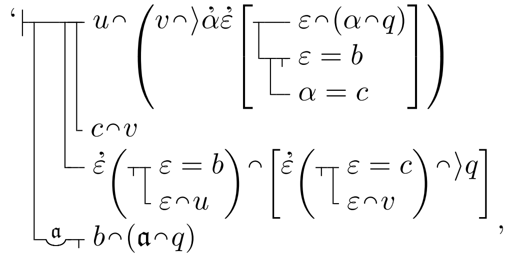
(α

| For ease of understanding it is more convenient to consider the proposition 102
resulting from the latter by contraposition:



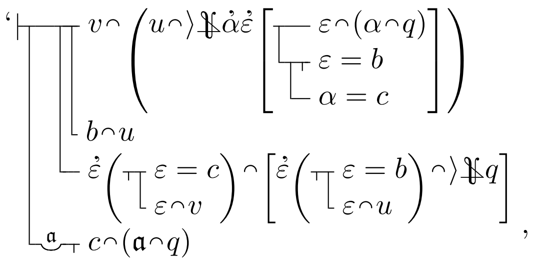
(β

In order to prove that the cardinal number of the u-concept is equal to the cardinal number of the v-concept, it will be sufficient, according to (32), to supply any relation which maps the u-concept into the v-concept and whose converse maps the latter into the former. We have already encountered such a relation in §66. Accordingly, we will first derive the proposition



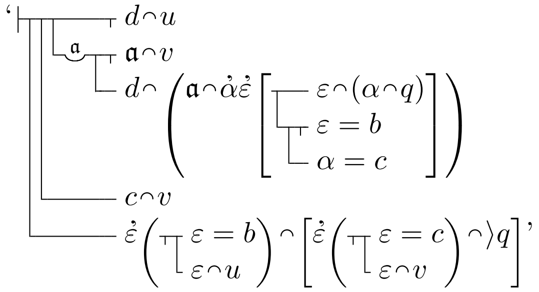
(γ

and then the proposition



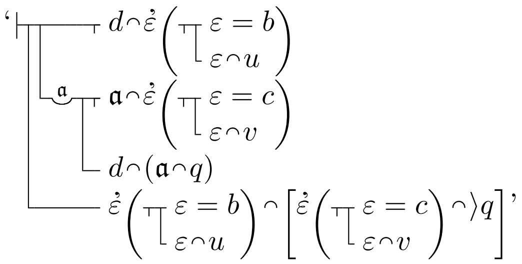
(δ

To prove the latter using (11), we first require the proposition



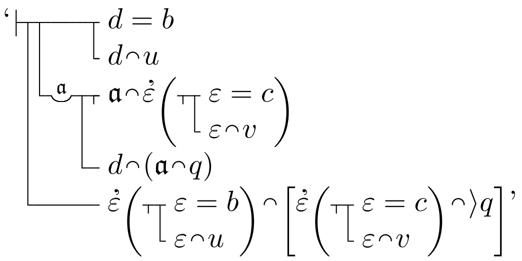
(ε

| The cases $d = b$ and $\top d = b$ are to be distinguished. We write (8) in the form 103



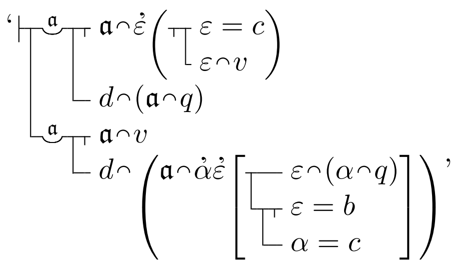
(ζ

from which straightforwardly follows



(η

| From this we easily arrive at our proposition for the case $\top d = b$ by contraposition, once we have proven 103a



(ϑ

If, for this purpose, we write (II a) in the form

$$`\vdash\begin{array}{l} a\frown v \\ d\frown\left(a\frown\dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l}\varepsilon\frown(\alpha\frown q)\\ \varepsilon=b\\ \alpha=c\end{array}\right]\right) \\ \overset{\mathfrak{a}}{\smile}\ \mathfrak{a}\frown v \\ d\frown\left(\mathfrak{a}\frown\dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l}\varepsilon\frown(\alpha\frown q)\\ \varepsilon=b\\ \alpha=c\end{array}\right]\right)\end{array}, \qquad (\iota$$

then we have to prove | 103b

$$`\vdash\begin{array}{l} d\frown\left(a\frown\dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l}\varepsilon\frown(\alpha\frown q)\\ \varepsilon=b\\ \alpha=c\end{array}\right]\right) \\ d\frown(a\frown q)\end{array}, \qquad (\kappa$$

which is straightforward using (I) and (36). Now

$$`\vdash\begin{array}{l} \top\, a\frown\dot{\varepsilon}\left(\begin{array}{l}\top\,\varepsilon=c\\ \varepsilon\frown v\end{array}\right) \\ \top\, a\frown v\end{array}, \qquad (\lambda$$

still has to be derived from (I a) in the form

$$`\vdash\begin{array}{l} \top\, a=c \\ a\frown v \\ \top\, a\frown v\end{array}, \qquad (\mu$$

and the proposition

$$`\vdash\begin{array}{l} F(\top\, a\frown\dot{\varepsilon}(\top\, f(\varepsilon))) \\ F(\text{—}\, f(a))\end{array}, \qquad (\nu$$

which follows from

$$`\vdash (\text{—}\, f(a)) = (\top\, a\frown\dot{\varepsilon}(\top\, f(\varepsilon)))' \qquad (\xi$$

This last proposition is straightforwardly proven from (1) using (IV b).

§79. *Construction* 104

[proof to be added]

We now prove the proposition

$$
\text{‘}\vdash\!\!\begin{array}{l}
\begin{array}{l} c\frown v \\ d = b \end{array} \\
\smile^{\mathfrak{a}}\begin{array}{l} \mathfrak{a}\frown v \\ d\frown\left(\mathfrak{a}\frown\grave{\alpha}\grave{\varepsilon}\left[\begin{array}{l} \varepsilon\frown(\alpha\frown q) \\ \begin{array}{l} \varepsilon = b \\ \alpha = c \end{array} \end{array}\right]\right) \end{array}
\end{array}\text{’} \qquad (\alpha
$$

which in conjunction with (60) leads to the proposition (ε) §78. If we write (II a) in the form

$$
\text{‘}\vdash\!\!\begin{array}{l}
\begin{array}{l} c\frown v \\ d\frown\left(c\frown\grave{\alpha}\grave{\varepsilon}\left[\begin{array}{l} \varepsilon\frown(\alpha\frown q) \\ \begin{array}{l} \varepsilon = b \\ \alpha = c \end{array} \end{array}\right]\right) \end{array} \\
\smile^{\mathfrak{a}}\begin{array}{l} \mathfrak{a}\frown v \\ d\frown\left(\mathfrak{a}\frown\grave{\alpha}\grave{\varepsilon}\left[\begin{array}{l} \varepsilon\frown(\alpha\frown q) \\ \begin{array}{l} \varepsilon = b \\ \alpha = c \end{array} \end{array}\right]\right) \end{array}
\end{array}\text{’} \qquad (\beta
$$

then

$$
\text{‘}\vdash\!\!\begin{array}{l}
d\frown\left(c\frown\grave{\alpha}\grave{\varepsilon}\left[\begin{array}{l} \varepsilon\frown(\alpha\frown q) \\ \begin{array}{l} \varepsilon = b \\ \alpha = c \end{array} \end{array}\right]\right) \\
d = b
\end{array}\text{’} \qquad (\gamma
$$

is still to be proven, which follows, using (36), from

$$
\text{‘}\vdash\!\!\begin{array}{l}
d\frown(c\frown q) \\
\begin{array}{l} d = b \\ c = c \end{array} \\
d = b
\end{array}\text{’} \qquad (\delta
$$

This proposition results from (I a) in the form

$$
\text{‘}\vdash\!\!\begin{array}{l}
d\frown(c\frown q) \\
d = b \\
d = b
\end{array}\text{’}
$$

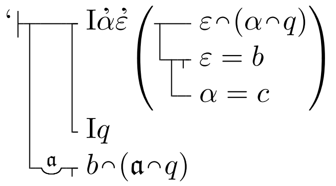
and (I) and (III e).

§81. *Construction* 105b

[proof to be added]

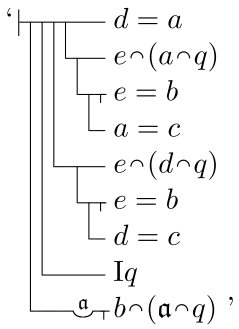
§82. *Analysis* 106a

We lack a proof of the proposition



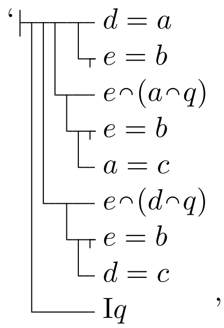
(α

(compare §78, γ). In order to conduct it using (34) we need the proposition



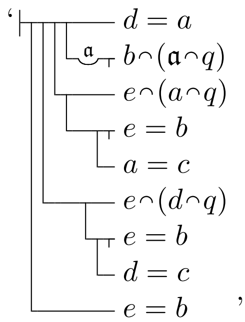
(β

which we may derive from the propositions | 106b



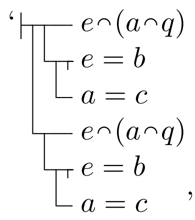
(γ

and

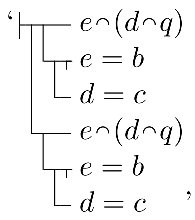


(δ

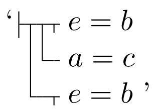
by rule (8). (γ) follows from (13) using (I) in the forms | 107a



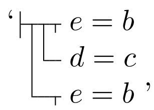
and



The subcomponents ‘$\top\, e = b$, $a = c$’ and ‘$\top\, e = b$, $d = c$’ initially occur here. These can be replaced by ‘$\top\, e = b$’, using (I) in the forms



and

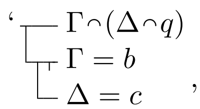


§83. *Construction*

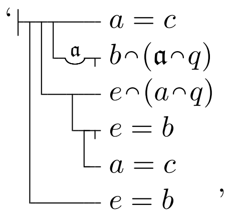
[proof to be added]

§84. *Analysis* 107b

In order to prove the proposition (δ) of §82, we now note that

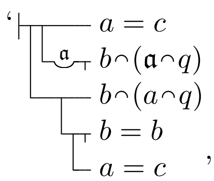


indicates the truth-value of: that Γ stands in the q-relation to Δ, or that Γ coincides with b and Δ with c. If we now take Γ for b, then only the latter of the two cases can occur if there is an object to which b stands in the q-relation; i.e., Δ must then coincide with c. Accordingly, it will be possible to prove the proposition,



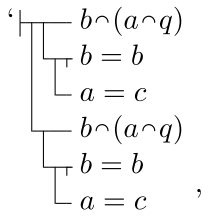
(α

which, in a first step, follows from



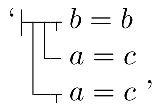
(β

If we now write (I) in the form

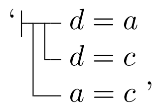


(γ

we can then apply (I a) to it in the form



| and then, by contraposition and (II a), easily arrive at our proposition (β). 108a
We then replace 'a' by 'd' in the proposition (α) and, using (III a) in the form,



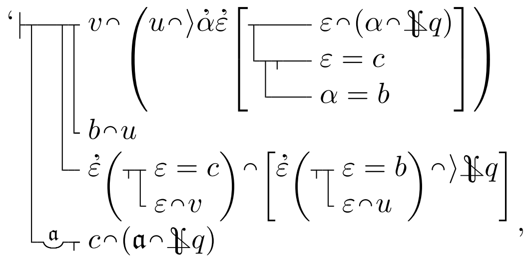
we obtain our goal.

§85. *Construction*

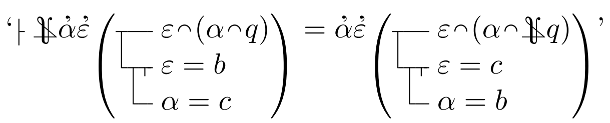
[proof to be added]

§86. *Analysis* 109b

We have thus proven the proposition (γ) of §78. In order to derive (δ), we exchange 'q' with '$\rceil\!\!\!\!\setminus q$', '$b$' with '$c$', '$u$' with '$v$' in (63). We thus get (63) in the form



and merely have to supply the proof of the proposition | 109



which is similar to that of §75 (κ).

§87. *Construction*

[proof to be added]

Γ. Proof of the proposition

$$`\vdash \mathrm{I}\!\not\between\!\mathfrak{f}\,'$$

a) Proof of the proposition

$$`\vdash\!\!\begin{array}{l} \dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = m \\ \llcorner\, \varepsilon \frown \dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = b\\ \llcorner\, \varepsilon \frown u\end{array}\right)\end{array}\right) \frown \left(\dot{\varepsilon}\left[\begin{array}{l}\top\, \varepsilon = n \\ \llcorner\, \varepsilon \frown \dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = c\\ \llcorner\, \varepsilon \frown v\end{array}\right)\end{array}\right] \frown\rangle \dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l}\top\!\top\, \varepsilon \frown (\alpha \frown q) \\ \;\llcorner\!\top\, \varepsilon = b \\ \;\;\llcorner\!\top\, \alpha = c\end{array}\right]\right) \\ \llcorner\, c \frown (m \frown \not\between q) \\ \llcorner\, \mathrm{I}\!\not\between q \\ \llcorner\, b \frown (n \frown q) \\ \llcorner\, u \frown (v \frown\rangle q) \end{array}\;'$$

§88. *Analysis* 113a

We now want to prove the proposition that, for every cardinal number, there is no more than one that immediately precedes it in the cardinal-number series. This takes us back to the proposition | 113b

$$`\vdash\!\!\begin{array}{l} d = a \\ \llcorner\, a \frown (e \frown \mathfrak{f}) \\ \llcorner\, d \frown (e \frown \mathfrak{f})\;' \end{array} \qquad (\alpha$$

If we here introduce the expressions resulting according to definition (H), then we have | 114a

$$`\vdash\!\!\begin{array}{l} d = a \\ \llcorner\!\top\!\overset{\mathfrak{u}}{\smile}\!\overset{\mathfrak{a}}{\smile}\!\top\!\top\, \wp\mathfrak{u} = e \\ \qquad\qquad \llcorner\, \mathfrak{a} \frown \mathfrak{u} \\ \qquad\qquad \llcorner\, \wp\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = \mathfrak{a}\\ \llcorner\, \varepsilon \frown \mathfrak{u}\end{array}\right) = a \\ \llcorner\!\top\!\overset{\mathfrak{u}}{\smile}\!\overset{\mathfrak{a}}{\smile}\!\top\!\top\, \wp\mathfrak{u} = e \\ \qquad\qquad \llcorner\, \mathfrak{a} \frown \mathfrak{u} \\ \qquad\qquad \llcorner\, \wp\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = \mathfrak{a}\\ \llcorner\, \varepsilon \frown \mathfrak{u}\end{array}\right) = d\;' \end{array} \qquad (\beta$$

a proposition which is obtained by repeated applications of contraposition and introduction of German letters from

$$
\text{‘}\vdash\!\!\begin{array}{l}
d = a \\
\quad \llcorner\ \not{p}u = e \\
\quad \llcorner\ b \frown u \\
\quad \llcorner\ \not{p}\dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = b \\ \quad \llcorner\ \varepsilon \frown u \end{array}\right) = d \\
\quad \llcorner\ \not{p}v = e \\
\quad \llcorner\ c \frown v \\
\quad \llcorner\ \not{p}\dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = c \\ \quad \llcorner\ \varepsilon \frown v \end{array}\right) = a \text{’}
\end{array} \qquad (\gamma
$$

This proposition can be derived from

$$
\text{‘}\vdash\!\!\begin{array}{l}
\not{p}\dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = b \\ \quad \llcorner\ \varepsilon \frown u \end{array}\right) = \not{p}\dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = c \\ \quad \llcorner\ \varepsilon \frown v \end{array}\right) \\
\quad \llcorner\ b \frown u \\
\quad \llcorner\ c \frown v \\
\quad \llcorner\ \not{p}u = \not{p}v
\end{array} \text{’} \qquad (\delta
$$

According to proposition (32), we now have only to disclose a relation that maps the $\dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = b \\ \quad \llcorner\ \varepsilon \frown u \end{array}\right)$-concept into the | $\dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = c \\ \quad \llcorner\ \varepsilon \frown v \end{array}\right)$-concept and whose con- 114b
verse maps the latter into the former. The subcomponent ‘$\not{p}u = \not{p}v$’ tells us that there is a relation which maps the u-concept into the v-concept and whose converse maps the latter into the former. Let the q-relation be such a relation. We now know of the $\dot{\alpha}\dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\!\!\top\ \varepsilon \frown (\alpha \frown q) \\ \quad \llcorner\!\top\ \varepsilon = b \\ \qquad \llcorner\!\top\ \alpha = c \end{array}\right)$-relation that no object stands to c, and b stands to no object in this relation.[1] Moreover, no objects stands to n and m stands to no object in this relation if m stands to c and b to n in the q-relation, since the latter, like its converse, is single-valued. The former relation maps the $\dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = m \\ \quad \llcorner\ \varepsilon \frown \dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = b \\ \quad \llcorner\ \varepsilon \frown u \end{array}\right) \end{array}\right)$-concept into the $\dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = n \\ \quad \llcorner\ \varepsilon \frown \dot{\varepsilon}\left(\begin{array}{l} \top\!\!\top\ \varepsilon = c \\ \quad \llcorner\ \varepsilon \frown v \end{array}\right) \end{array}\right)$-concept, and its converse maps the latter into the former. By (32) the cardinal number of the latter concept is equal to the cardinal number of the former. Using (66) we can then hope to reach our goal.

First, we turn to the proof of the proposition

[1] Compare §66.

$$‘\vdash\begin{array}{l}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon = m\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=b\\ \varepsilon\frown u\end{array}\right)\end{array}\right)\frown\left(\dot{\varepsilon}\left[\begin{array}{l}\varepsilon = n\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=c\\ \varepsilon\frown v\end{array}\right)\end{array}\right]\frown\rangle\dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l}\varepsilon\frown(\alpha\frown q)\\ \varepsilon=b\\ \alpha=c\end{array}\right]\right)\\ c\frown(m\frown\not\rangle q)\\ \mathrm{I}\!\!\not\rangle q\\ b\frown(n\frown q)\\ u\frown(v\frown\rangle q)\end{array}’\qquad(\varepsilon$$

| If we write (51) in the form 115

$$‘\vdash\begin{array}{l}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon = m\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=b\\ \varepsilon\frown u\end{array}\right)\end{array}\right)\frown\left(\dot{\varepsilon}\left[\begin{array}{l}\varepsilon = n\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=c\\ \varepsilon\frown v\end{array}\right)\end{array}\right]\frown\rangle\dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l}\varepsilon\frown(\alpha\frown q)\\ \varepsilon=b\\ \alpha=c\end{array}\right]\right)\\ b\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon = m\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=b\\ \varepsilon\frown u\end{array}\right)\end{array}\right)\\ c\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon = n\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=c\\ \varepsilon\frown v\end{array}\right)\end{array}\right)\\ \dot{\varepsilon}\left(\begin{array}{l}\varepsilon = m\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=b\\ \varepsilon\frown u\end{array}\right)\end{array}\right)\frown\left(\varepsilon\left[\begin{array}{l}\varepsilon = n\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=c\\ \varepsilon\frown v\end{array}\right)\end{array}\right]\frown\rangle q\right)\end{array}’,$$

then we see that what primarily remains to be proven is

$$‘\vdash\begin{array}{l}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon = m\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=b\\ \varepsilon\frown u\end{array}\right)\end{array}\right)\frown\left(\dot{\varepsilon}\left[\begin{array}{l}\varepsilon = n\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=c\\ \varepsilon\frown v\end{array}\right)\end{array}\right]\frown\rangle q\right)\\ m\frown(c\frown q)\\ \mathrm{I}\!\!\not\rangle q\\ b\frown(n\frown q)\\ u\frown(v\frown\rangle q)\end{array}’,\qquad(\zeta$$

because the proposition

$$‘\vdash b\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon = m\\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=b\\ \varepsilon\frown u\end{array}\right)\end{array}\right)’$$

presents no difficulty. Once we have proven the proposition

$$‘\vdash\begin{array}{l}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=b\\ \varepsilon\frown u\end{array}\right)\frown\left[\dot{\varepsilon}\left(\begin{array}{l}\varepsilon=n\\ \varepsilon\frown v\end{array}\right)\frown\rangle q\right]\\ b\frown(n\frown q)\\ \mathrm{I}\!\!\not\rangle q\\ u\frown(v\frown\rangle q)\end{array}’,\qquad(\eta$$

then we can apply it twice and thereby arrive at the proposition

$$‘\vdash\!\!\begin{array}{l}\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = m\\ \llcorner\, \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = b\\ \llcorner\, \varepsilon\frown u\end{array}\right)\end{array}\right)\frown\left(\dot{\varepsilon}\left[\begin{array}{l}\top\, \varepsilon = c\\ \llcorner\, \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = n\\ \llcorner\, \varepsilon\frown v\end{array}\right)\end{array}\right]\frown\rangle q\right)\\ \llcorner\, m\frown(c\frown q)\\ \llcorner\, \mathrm{I}\!\!\not\rangle q\\ \llcorner\, b\frown(n\frown q)\\ \llcorner\, u\frown(v\frown\rangle q)\end{array}\quad ’ \qquad (\vartheta$$

We now still require the propositions

$$‘\vdash \dot{\varepsilon}g(\varepsilon, f(\varepsilon)) = \dot{\varepsilon}g(\varepsilon, \varepsilon\frown\dot{\varepsilon}f(\varepsilon))\,’ \qquad (\iota$$

| and 116a

$$‘\vdash \dot{\varepsilon}\left(\begin{array}{l}\top\!\!-\, f(\varepsilon)\\ \llcorner\top\, g(\varepsilon)\\ \quad\llcorner\, h(\varepsilon)\end{array}\right) = \dot{\varepsilon}\left(\begin{array}{l}\top\!\!-\, g(\varepsilon)\\ \llcorner\top\, f(\varepsilon)\\ \quad\llcorner\, h(\varepsilon)\end{array}\right)’ \qquad (\kappa$$

in order to be able to replace

$$‘\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = c\\ \llcorner\, \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = n\\ \llcorner\, \varepsilon\frown v\end{array}\right)\end{array}\right)’$$

by

$$‘\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = n\\ \llcorner\, \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = c\\ \llcorner\, \varepsilon\frown v\end{array}\right)\end{array}\right)’$$

These propositions will be derived first.

§89. *Construction*

[proof to be added]

§90. *Analysis*

We now prove proposition (η) of §88. If the q-relation maps the u-concept into the v-concept, then there is, for every object falling under the u-concept, one falling under the v-concept to which it stands in the q-relation. Now, every object falling under the $\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = b\\ \llcorner\, \varepsilon\frown u\end{array}\right)$-concept falls under the u-concept, and thus, given our condition, there is, for every object falling under the $\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = b\\ \llcorner\, \varepsilon\frown u\end{array}\right)$-concept, one falling under the v-concept to which the former
stands in the q-relation; yet | n falls under the v-concept but does not fall 117b
under the $\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon = n\\ \llcorner\, \varepsilon\frown v\end{array}\right)$-concept. Now, if an object which stood to n in the

q-relation fell under the $\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon = b\\ \llcorner\ \varepsilon \frown u\end{array}\right)$-concept, then there would not have to be an object to which it stood in the q-relation and which fell under the $\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon = n\\ \llcorner\ \varepsilon \frown v\end{array}\right)$-concept. This case is excluded, however, by the subcomponents '$—\, b\frown(n\frown q)$' and '$\mathrm{I}\rangle\!\!\!\not\,q$'.

§91. *Construction*

[proof to be added]

b) Proof of the proposition

$$
\text{`}\begin{array}{l}
\vdash\!\top\!\top\ \mathfrak{p}u = \mathfrak{p}v\\
\quad\llcorner\!\top\ \mathfrak{p}\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon = b\\ \llcorner\ \varepsilon \frown u\end{array}\right) = \mathfrak{p}\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon = c\\ \llcorner\ \varepsilon \frown v\end{array}\right)\\
\quad\llcorner\ b\frown u\\
\quad\llcorner\ c\frown v
\end{array}\text{',}
$$

and end of section Γ

§92. *Analysis* 121

From (83) we can derive, using (53) and (22), the proposition

$$
\text{`}\begin{array}{l}
\vdash\!\top\!\top\!\top\!\top\ \dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon = n\\ \llcorner\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon = c\\ \llcorner\ \varepsilon\frown v\end{array}\right)\end{array}\right) \frown \left(\dot{\varepsilon}\left[\begin{array}{l}\top\ \varepsilon = m\\ \llcorner\ \varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon = b\\ \llcorner\ \varepsilon\frown u\end{array}\right)\end{array}\right] \frown\rangle\!\not\,\dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l}\top\!\top\ \varepsilon\frown(\alpha\frown q)\\ \quad\llcorner\!\top\ \varepsilon = b\\ \quad\quad\llcorner\!\top\ \alpha = c\end{array}\right]\right)\\
\quad\llcorner\ b\frown(n\frown q)\\
\quad\llcorner\ \mathrm{I}\rangle\!\!\not\,\rangle\!\!\not\,q\\
\quad\llcorner\ c\frown(m\frown\rangle\!\!\not\,q)\\
\quad\llcorner\ v\frown(u\frown\rangle\!\!\not\,q)
\end{array}\quad(\alpha\quad\text{',}
$$

From this proposition and (84), we arrive, using (66), at a proposition with the supercomponent

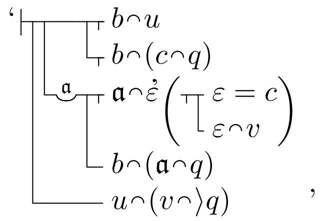
$$
\text{`}\ \mathfrak{p}\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon = b\\ \llcorner\ \varepsilon\frown u\end{array}\right) = \mathfrak{p}\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon = c\\ \llcorner\ \varepsilon\frown v\end{array}\right)\text{'}
$$

§93. *Construction*

[proof to be added]

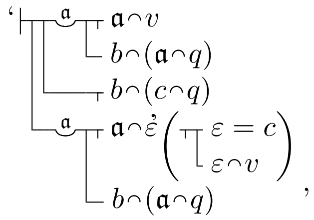
§94. *Analysis* 123a

The preceding two transitions already point to the path that is to be followed from here on. In (ε) we had n and m as auxiliary objects similar to auxiliary lines in geometry. These should not appear in our proposition and so must be removed. This is achieved, as always, by showing that there is something with the required constitution, or, more conveniently in concept-script, that if there were nothing of this kind, then one of the assumptions we make would not hold. | Our task is now to prove the proposition 123b

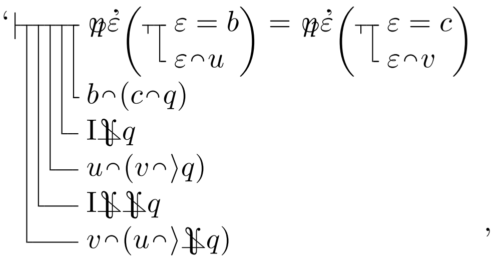


(α

Hereby we add the subcomponent ‘$\top b\frown(c\frown q)$’; but we can also prove the otherwise unaltered proposition with the contrary subcomponent ‘— $b\frown(c\frown q)$’. In order to derive (α) | from (8) we require the proposition 124a



which follows from (77). Like ‘n’, we remove ‘m’ from our proposition and then, as was just indicated, prove the proposition



using (80) and (32), and eliminate the subcomponents ‘— $b\frown(c\frown q)$’ and ‘$\top b\frown(c\frown q)$’ by rule (8). After removing the subcomponents ‘I⅄q’ and ‘I⅄⅄q’ using (30) and (18), we employ (49) to attain the goal of section (b), and thereafter, as was indicated in §88, arrive at the proposition ‘⊦ I⅄f’ by means of (68).

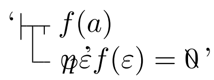
§95. *Construction*

[proof to be added]

127

Δ. Proofs of some propositions concerning the cardinal number Zero

a) *Proof of the proposition*



§96. *Analysis* 127a

We now prove the proposition that no object falls under a concept whose associated cardinal number is $\emptyset$. | The proposition mentioned in the heading 127b is actually somewhat more general, because the function-letter 'f' does not only indicate concepts. | Our proposition follows straightforwardly from the 128a proposition

$$`\begin{array}{l}\vdash\!\!\top\ \mathfrak{p}u = \emptyset \\ \ \ \llcorner\ a \frown u\end{array},$$

According to definition (Θ) we need to prove

$$`\begin{array}{l}\vdash\!\!\top\ \mathfrak{p}u = \mathfrak{p}\dot{\varepsilon}(\top\, \varepsilon = \varepsilon) \\ \ \ \llcorner\ a \frown u\end{array},$$

which can be achieved by means of (49), by proving $`\begin{array}{l}\vdash\!\!\top\ a \frown u \\ \ \ \llcorner\ u \frown (\dot{\varepsilon}(\top\, \varepsilon = \varepsilon) \frown \rangle q)\end{array}$'.

This requires use of (8) and the proposition $`\begin{array}{l}\vdash\!\!\top\ b \frown \dot{\varepsilon}(\top\, \varepsilon = \varepsilon) \\ \ \ \llcorner\ a \frown (b \frown q)\end{array}$' which is straightforwardly obtained from (58).

§97. *Construction*

[proof to be added]

b) Proof of the proposition

$$`\begin{array}{l}\vdash\!\!\!-\!\!-\!\!-\ \mathfrak{p}u = \emptyset \\ \ \ \llcorner\!\overset{\mathfrak{a}}{\smile}\!\top\ \mathfrak{a} \frown u\end{array},$$

and of some correlaries

§98. *Analysis*

The proposition mentioned in the heading is somewhat more general than the one we express in words like this: "If no object falls under a concept, then Zero is the cardinal number that belongs to this concept."

| We first prove, using (32) and (38), the proposition 128b

$$`\begin{array}{l}\vdash\!\!\!-\!\!-\!\!-\ \mathfrak{p}u = \mathfrak{p}v \\ \ \ \llcorner\!\overset{\mathfrak{a}}{\smile}\ (-\ \mathfrak{a} \frown u) = (-\ \mathfrak{a} \frown v)\end{array}'$$

and then

$$`\begin{array}{l}\vdash\!\!\!-\!\!-\!\!-\ (-\ a \frown u) = (-\ a \frown \dot{\varepsilon}(\top\, \varepsilon = \varepsilon)) \\ \ \ \llcorner\!\overset{\mathfrak{a}}{\smile}\!\top\ \mathfrak{a} \frown u\end{array},$$

§99. *Construction* 129a

[proof to be added]

§100. *Analysis* 129b

We begin by deriving some straightforward consequences from (97), and then turn our attention to the proposition: "If a cardinal number is not Zero, then there is one that immediately precedes it in the cardinal number series", in signs:

$$`\vdash \begin{array}{l} \neg\forall\mathfrak{a}\,\neg\, \mathfrak{a}\frown(a\frown\mathfrak{f}) \\ \neg\, a = \emptyset \\ \neg\forall\mathfrak{u}\,\neg\, \wp\mathfrak{u} = a \end{array}\text{'},$$

We first derive the simpler proposition

$$`\vdash \begin{array}{l} \neg\, c\frown u \\ \neg\forall\mathfrak{a}\,\neg\, \mathfrak{a}\frown(\wp u\frown\mathfrak{f}) \end{array}\text{'}$$

For this we require the proposition

$$`\vdash \begin{array}{l} \wp\grave{\varepsilon}\left(\begin{array}{l}\neg\, \varepsilon = c \\ \varepsilon\frown u\end{array}\right)\frown(\wp u\frown\mathfrak{f}) \\ c\frown u \end{array}\text{'},$$

which follows from definition (H).

§101. *Construction*

[proof to be added]

E. Proofs of some propositions concerning the cardinal number One

§102. *Analysis* 131a

We prove the proposition

$$`\vdash \begin{array}{l} \neg\forall\mathfrak{a}\,\neg\, \mathfrak{a}\frown u \\ \wp u = \mathbb{1} \end{array}\text{'} \qquad (\alpha$$

which may be expressed in words like this:

"There is an object which falls under a concept, if One is the cardinal number of this concept."

If this were not correct, then according to proposition (97) the cardinal number One would coincide with the cardinal number Zero. What needs to be shown is that this cannot be. To this end we prove the propositions

$$`\vdash \emptyset\frown(\mathbb{1}\frown\mathfrak{f})\text{'} \quad (\beta, \qquad `\vdash \neg\, \emptyset\frown(\emptyset\frown\mathfrak{f})\text{'} \quad (\gamma$$

Of these, (β) follows from (101) by definition (I), (γ) from (68) using proposition (93).

§103. *Construction*

[proof to be added]

§104. *Analysis* 132a

Using (110) and (71) it is straightforward to prove the proposition that a cardinal number is One if it immediately follows Zero in the cardinal number series.

In order to prove the proposition

$$‘\vdash \begin{array}{l} d = a \\ a \frown u \\ \natural u = \mathbb{1} \\ d \frown u \end{array} ’ \qquad (\alpha$$

| we apply (49) in the form 132b

$$‘\vdash \begin{array}{l} \natural u = \natural \dot{\varepsilon}(\varepsilon = \mathbb{0}) \\ \smile^{\mathfrak{q}}\; u \frown (\dot{\varepsilon}(\varepsilon = \mathbb{0}) \frown \rangle \mathfrak{q}) \\ \dot{\varepsilon}(\varepsilon = \mathbb{0}) \frown (u \frown \rangle \rangle\!\!\!\!\!\!\; \mathfrak{q}) \end{array} ’$$

and now require the proposition

$$‘\vdash \begin{array}{l} d = a \\ \dot{\varepsilon}(\varepsilon = \mathbb{0}) \frown (u \frown \rangle \rangle\!\!\!\!\!\!\; q) \\ u \frown (\dot{\varepsilon}(\varepsilon = \mathbb{0}) \frown \rangle q) \\ a \frown u \\ d \frown u \end{array} ’ \qquad (\beta$$

From (79) and (18) we have the proposition

$$‘\vdash \begin{array}{l} d = a \\ a \frown (\mathbb{0} \frown q) \\ d \frown (\mathbb{0} \frown q) \\ \dot{\varepsilon}(\varepsilon = \mathbb{0}) \frown (u \frown \rangle \rangle\!\!\!\!\!\!\; q) \end{array} ’ \qquad (\gamma$$

and now apply the proposition

$$‘\vdash \begin{array}{l} a \frown (c \frown q) \\ a \frown u \\ u \frown (\dot{\varepsilon}(\varepsilon = c) \frown \rangle q) \end{array} ’$$

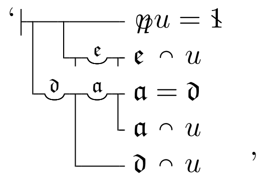
which is straightforwardly derived using (77) and (8).

§105. *Construction*

[proof to be added]

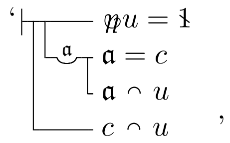
§106. *Analysis* 133b

We now prove the proposition

 (α

i.e., "One is the cardinal number of a concept under which an object falls if, whenever an object a and an object d fall under that concept, it follows that a is the same as d."

This proposition is a consequence of the proposition

 (β

which we trace back to the following proposition

$$\text{‘}\vdash \mathfrak{1} = \mathfrak{p}\dot{\varepsilon}(\varepsilon = c)\text{’} \quad \text{or}$$

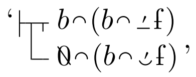
$$\text{‘}\vdash \mathfrak{p}\dot{\varepsilon}(\varepsilon = n) = \mathfrak{p}\dot{\varepsilon}(\varepsilon = c)\text{’} \qquad (\gamma$$

The $\dot{\alpha}\dot{\varepsilon}\left(\begin{smallmatrix}\alpha = c\\ \varepsilon = n\end{smallmatrix}\right)$-relation suggests itself as a mapping relation. | Its single- 134a
valuedness, and that of its converse, has to be proven.

§107. *Construction*

[proof to be added]

Z. Proof of the proposition 137



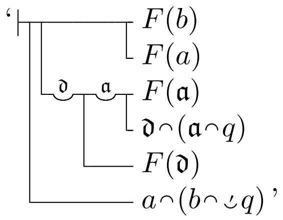
a) Proof of the proposition

$$\text{‘}\vdash a \frown (\emptyset \frown \underline{\smile}\mathfrak{f})\text{’}$$

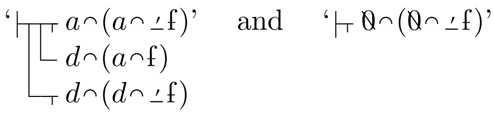
§108. *Analysis* 137a

The proposition mentioned in the main heading states that no object belonging to the cardinal number series starting with Zero follows after itself in the cardinal number series. Instead, we could also say: "No *finite* cardinal number follows after itself in the cardinal number series." The importance

of this proposition may be brought out more clearly through the following consideration. When we determine the cardinal number belonging to a concept $\Phi(\xi)$, or, as one normally says, when we count the objects falling under a concept $\Phi(\xi)$, we correlate these with the numerals, one after the other, beginning with 'One' up to that numeral 'N' which is determined by the correlating relation mapping the concept $\Phi(\xi)$ into the concept "member of the series of numerals from 'One' to 'N'" and the converse relation mapping the latter into the former. 'N' then designates the desired cardinal number; i.e., N is that cardinal number. This process allows of manifold execution since the correlating relation is not completely specified. The question arises whether a different numeral 'M' could have been reached under a different choice of relation. According to our specifications, M would then be | the 137b
same cardinal number as N; at the same time, however, one of the numerals would follow the other, e.g., 'N' would follow 'M'. In that case, N would also follow M in the cardinal number series, that is, it would follow itself. This is excluded for finite cardinal numbers by our proposition. We prove it using the propositions



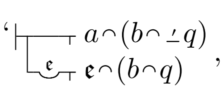
(α



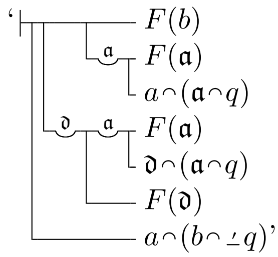
The last one is a special case of

'$\vdash a^\frown(0^\frown\,\underline{\smile}\,\mathfrak{f})$'

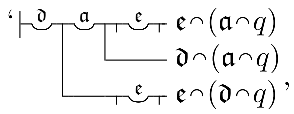
which says that the cardinal number 0 does not follow any object in the cardinal number series. We prove this proposition first. For this we need the proposition



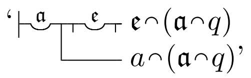
and (108). The former says that an object follows no object in the q-series if no object stands to it in the q-re|lation. In order to prove this we require 138a the proposition



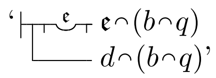
which follows from (K) and (6). We then replace the function-marker ‘$F(\xi)$’ by ‘$\frown\!\!\mathfrak{e}\!\!\frown$ $e\frown(\xi\frown q)$’ and then have to prove the propositions



and



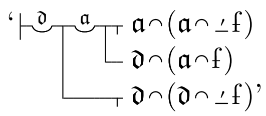
both of which follow from



§109. *Construction*

[proof to be added]

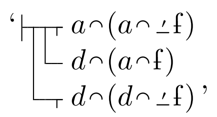
b) Proof of the proposition 139



and end of section Z

§110. *Analysis* 139a

The proposition



(α

is obtained by contraposition from

$$‘\vdash\begin{array}{l} d\frown(d\frown\underline{\cdot}\,\mathfrak{f}) \\ d\frown(a\frown\mathfrak{f}) \\ a\frown(a\frown\underline{\cdot}\,\mathfrak{f})’ \end{array} \qquad (\beta$$

This proposition can be concluded from the propositions

$$‘\vdash\begin{array}{l} a\frown(d\frown\dot{\smile}\,\mathfrak{f}) \\ a\frown(a\frown\underline{\cdot}\,\mathfrak{f}) \\ d\frown(a\frown\mathfrak{f})’ \end{array} \qquad (\gamma$$

and

$$‘\vdash\begin{array}{l} d\frown(c\frown\underline{\cdot}\,q) \\ d\frown(a\frown q) \\ a\frown(c\frown\dot{\smile}\,q)’ \end{array} \qquad (\delta$$

by replacing ‘c’ by ‘d’ and ‘q’ by ‘$\mathfrak{f}$’ in the latter. We prove (δ) from the propositions

$$‘\vdash\begin{array}{l} F\left(\begin{array}{l} c = a \\ \neg\, a\frown(c\frown\underline{\cdot}\,q) \end{array}\right) \\ F(a\frown(c\frown\dot{\smile}\,q))’, \end{array} \qquad (\varepsilon$$

$$‘\vdash\begin{array}{l} d\frown(c\frown\underline{\cdot}\,q) \\ d\frown(a\frown q) \\ c = a’, \end{array} \qquad (\zeta$$

and

$$‘\vdash\begin{array}{l} d\frown(c\frown\underline{\cdot}\,q) \\ d\frown(a\frown q) \\ a\frown(c\frown\underline{\cdot}\,q)’ \end{array} \qquad (\eta$$

| which follow straightforwardly from (Λ), (K), and (123). 139b

§111. *Construction*

[proof to be added]

140b

§112. *Analysis*

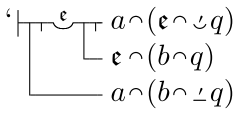
We now have to prove the proposition (γ) of §110. It is a special case of

$$‘\vdash\begin{array}{l} a\frown(d\frown\dot{\smile}\,\mathfrak{f}) \\ a\frown(b\frown\underline{\cdot}\,\mathfrak{f}) \\ d\frown(b\frown\mathfrak{f})’, \end{array}$$

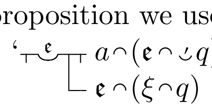
in words:

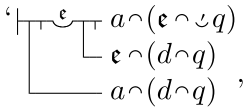
“If a cardinal number (b) follows a second cardinal number (a) in the cardinal number series, and immediately follows a third (d) in the cardinal number series, then the third (d) belongs to the cardinal number series starting with the second (a)."

Clearly, the analogue would not hold in general for an arbitrary series. It is essential here | that predecession in the cardinal number series is single- 141a
valued (88). We rely on the proposition that if an object (b) follows a second (a) in an arbitrary (q-)series then there is an object that belongs to the (q-)series which starts with the second (a) and which stands to the first in the series-forming (q-)relation; in signs:



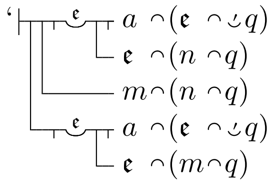
(α

If one now knows that there is no more than one object that stands in the (q-)relation to the first (b), then this must also belong to the (q-)series starting with the second (a). For the proof of this proposition we use (123) in which we replace the function-marker '$F(\xi)$' by ''. We thus require the two propositions



(β

and

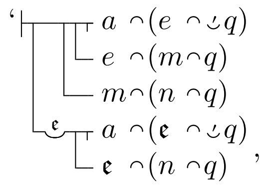


(γ

The former follows, using (II a), from the proposition

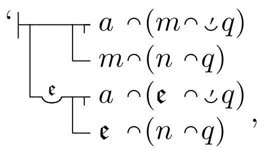
$$\text{'}\vdash a \frown (a \frown \smile q)\text{'}$$

which is a consequence of definition (Λ). The latter is obtained by introduction of the German '$\mathfrak{e}$' and contraposition from | 141b



(δ

By (II a) we now have



What remains to be shown is

$$‘\vdash\begin{array}{l}\neg\, a\frown(e\frown\acute{\smile}q)\\ \quad e\frown(m\frown q)\\ \neg\, a\frown(m\frown\acute{\smile}q)\end{array}’$$

or

$$‘\vdash\begin{array}{l}a\frown(m\frown\acute{\smile}q)\\ \quad e\frown(m\frown q)\\ a\frown(e\frown\acute{\smile}q)\end{array}’ \qquad (\varepsilon$$

(ε) is a consequence of

$$‘\vdash\begin{array}{l}a\frown(m\frown\acute{\smile}q)\\ a\frown(m\frown\acute{\bot}q)\end{array}’ \qquad (\zeta$$

and

$$‘\vdash\begin{array}{l}a\frown(m\frown\acute{\bot}q)\\ \quad e\frown(m\frown q)\\ a\frown(e\frown\acute{\smile}q)\end{array}’ \qquad (\eta$$

The latter proposition is to be proven in a manner similar to that of (132).

§113. *Construction*

[proof to be added]

144

H. Proof of the proposition

$$‘\vdash\begin{array}{l}b\frown\mathfrak{p}(b\frown\acute{\smile}\mathfrak{f})\frown\mathfrak{f}\\ \emptyset\frown(b\frown\acute{\smile}\mathfrak{f})\end{array}’$$

§114. *Analysis* 144a

We want to prove the proposition that the cardinal number that belongs to the concept

belonging to the cardinal number series ending with b

immediately follows b in the cardinal numbers series, provided b is a finite cardinal number. This has the immediate corollary that the cardinal number series is infinite; i.e., that for every finite cardinal number, there is one that immediately follows it.

We first attempt the proof with the proposition (144) by replacing the function-marker ‘$F(\xi)$’ by ‘$\xi\frown(\mathfrak{p}(\xi\frown\acute{\smile}\mathfrak{f})\frown\mathfrak{f})$’. For this we require the proposition

$$`\vdash\!\!\begin{array}{l} a\smallfrown(\mathfrak{p}(a\smallfrown\dot{\smallsmile}\mathfrak{f})\smallfrown\mathfrak{f}) \\ d\smallfrown(a\smallfrown\mathfrak{f}) \\ d\smallfrown(\mathfrak{p}(d\smallfrown\dot{\smallsmile}\mathfrak{f})\smallfrown\mathfrak{f})\,' \end{array} \quad (\alpha$$ [1]

If we put ‘$(a\smallfrown\dot{\smallsmile}\mathfrak{f})$’ for ‘$u$’, | ‘$a$’ for ‘$m$’ and for ‘$c$’ in (102), then we obtain 144b

$$`\vdash\!\!\begin{array}{l} a\smallfrown(\mathfrak{p}(a\smallfrown\dot{\smallsmile}\mathfrak{f})\smallfrown\mathfrak{f}) \\ a\smallfrown(a\smallfrown\dot{\smallsmile}\mathfrak{f}) \\ \mathfrak{p}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon = a \\ \varepsilon\smallfrown(a\smallfrown\dot{\smallsmile}\mathfrak{f})\end{array}\right) = a\,' \end{array}$$

from which we can remove the subcomponent

$$`\text{—}\; a\smallfrown(a\smallfrown\dot{\smallsmile}\mathfrak{f})'$$

using (140). The question is whether the subcomponent

$$`\text{—}\; \mathfrak{p}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon = a \\ \varepsilon\smallfrown(a\smallfrown\dot{\smallsmile}\mathfrak{f})\end{array}\right) = a\,'$$

can be established as a consequence of

$$`d\smallfrown(a\smallfrown\mathfrak{f})' \quad \text{and} \quad `d\smallfrown(\mathfrak{p}(d\smallfrown\dot{\smallsmile}\mathfrak{f})\smallfrown\mathfrak{f})'$$

Because of the single-valuedness of progression in the cardinal number series (70) we have

$$`\vdash\!\!\begin{array}{l} \mathfrak{p}(d\smallfrown\dot{\smallsmile}\mathfrak{f}) = a \\ d\smallfrown(a\smallfrown\mathfrak{f}) \\ d\smallfrown(\mathfrak{p}(d\smallfrown\dot{\smallsmile}\mathfrak{f})\smallfrown\mathfrak{f})\,' \end{array} \quad (\beta$$

We thus attempt to establish whether

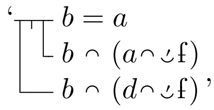
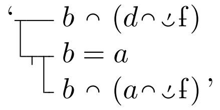
$$`\mathfrak{p}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon = a \\ \varepsilon\smallfrown(a\smallfrown\dot{\smallsmile}\mathfrak{f})\end{array}\right) = \mathfrak{p}(d\smallfrown\dot{\smallsmile}\mathfrak{f})\,'$$

| is a consequence of ‘$d\smallfrown(a\smallfrown\mathfrak{f})$’. This will requires (96). For this 145a

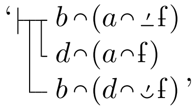
$$`\left(\begin{array}{l} b = a \\ b\smallfrown(a\smallfrown\dot{\smallsmile}\mathfrak{f})\end{array}\right) = (\text{—}\; b\smallfrown(d\smallfrown\dot{\smallsmile}\mathfrak{f}))\,'$$

has to be established as a consequence of ‘$d\smallfrown(a\smallfrown\mathfrak{f})$’, for which (IV a) will have to be used. It would thus have to be shown that the cardinal numbers belonging to the cardinal number series ending with a first cardinal number (a), except the latter itself, are the same as those that belong to the cardinal number series ending with a second cardinal number (d) if the first cardinal number (a) immediately follows a second cardinal number (d) in the cardinal number series. For this it is necessary to establish

[1] This proposition is, it seems, unprovable, but is not asserted as true here either, since it stands in quotation marks.

 and 

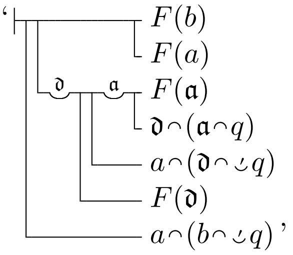
as consequences of ‘$d \frown (a \frown \mathfrak{f})$’. It turns out, however, that a further condition has to be added. Namely, ‘$\top\, b = a$’ would have to be shown to be a consequence of ‘$b \frown (d \frown \smile \mathfrak{f})$’ and ‘$d \frown (a \frown \mathfrak{f})$’. Now, according to (134) we have



If b and a were to coincide, then the supercomponent would turn into ‘— $a \frown (a \frown \underline{\smile} \mathfrak{f})$’. According to (145), this is excluded when a is a finite cardinal number. Thus the subcomponent

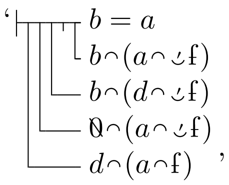
‘— $\mathfrak{d} \frown (a \frown \smile \mathfrak{f})$’

is added. Thereby, applying (144) in the way we had intended becomes of course impossible; however, using (137) we can replace this subcomponent | 145b
by ‘— $\mathfrak{d}(d \frown \smile \mathfrak{f})$’ and derive from (144) the proposition



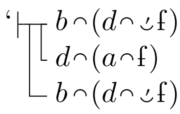
(γ

which then brings us to our goal. First, in order to have the proposition

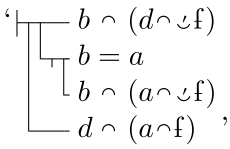


(δ

in full, we need to draw upon the proposition (137) in the form

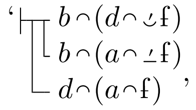


Then the proposition

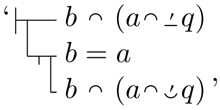


(ε

remains to be proven. According to (143) we have



For this, we now require the proposition

 (ζ

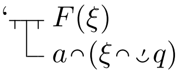
which follows straightforwardly from (130).

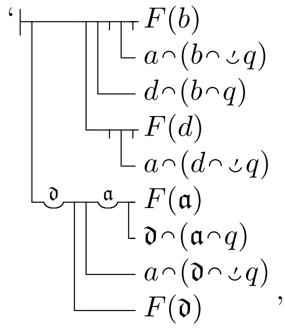
| **§115.** *Construction* 146a

[proof to be added]

147a

§116. *Analysis*

In order to prove the proposition (γ) of §114, we put ‘’ in place of the function-marker ‘$F(\xi)$’ in (144). We then have to prove | 147b



which can easily be done using (137). For the transition to (γ) compare p. 68.

§117. *Construction*

[proof to be added]

| **§118.** *Analysis* 149a

It now remains to prove the proposition

$$`\vdash \emptyset \frown (\not{\wp}(\emptyset \frown \smile\mathfrak{f}) \frown \mathfrak{f})\,'$$

By (102) we have

$$`\begin{array}{l} \vdash\!\!\top\!\top\, \emptyset \frown (\not{\wp}(\emptyset \frown \smile\mathfrak{f}) \frown \mathfrak{f}) \\ \quad\;\llcorner\, \emptyset \frown (\emptyset \frown \smile\mathfrak{f}) \\ \;\llcorner\, \not{\wp}\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = \emptyset \\ \quad\llcorner\, \varepsilon \frown (\emptyset \frown \smile\mathfrak{f})\end{array}\right) = \emptyset \end{array}\,'$$

Here we can use (140). We then still have to prove the proposition

$$`\vdash \not{\wp}\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = \emptyset \\ \quad\llcorner\, \varepsilon \frown (\emptyset \frown \smile\mathfrak{f})\end{array}\right) = \emptyset\,'$$

We apply the proposition (97) by showing that no object falls under the $\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = \emptyset \\ \quad\llcorner\, \varepsilon \frown (\emptyset \frown \smile\mathfrak{f})\end{array}\right)$-concept. This straightforwardly follows from

$$`\begin{array}{l}\vdash\!\!\top\, a = \emptyset \\ \quad\llcorner\, a \frown (\emptyset \frown \smile\mathfrak{f})\end{array}\,'$$

i.e., to the cardinal number series ending with $\emptyset$ only $\emptyset$ itself belongs. This proposition follows from (126) and (130).

| **§119.** *Construction* 149b

[proof to be added]

Θ. Some correlaries

| **§120.** *Analysis* 149a

First, what can easily be concluded from (155) is that for every finite cardinal number there is one immediately following it. Thereby it is stated that the cardinal number series starting with $\emptyset$ proceeds without end.

Moreover, we prove the proposition that provides a foundation for our counting by stating that n is the cardinal number belonging to a concept if a relation maps this concept into the car|dinal number series up to and includ- 149b
ing n and excluding $\emptyset$, and if the converse of this relation maps this cardinal number series into the concept, provided n is a finite cardinal number.

This proposition follows straightforwardly from the proposition

$$`\begin{array}{l}\vdash\!\!\top\, n = \not{\wp}\dot{\varepsilon}\left(\begin{array}{l}\top\!\top\, \varepsilon = \emptyset \\ \quad\llcorner\, \varepsilon \frown (n \frown \smile\mathfrak{f})\end{array}\right) \\ \quad\llcorner\, \emptyset \frown (n \frown \smile\mathfrak{f})\end{array}\,'$$

which we prove using (87) and (155).

| **§121.** *Construction* 150a

[proof to be added]

I. Proof of some propositions of the cardinal number Endlos

a) Proof of the proposition

‘$\vdash\!\top\ \emptyset\frown(\infty\frown\smile\mathfrak{f})$’

| **§122.** *Analysis* 150a

There are cardinal numbers that do not belong to the cardinal numbers series beginning with $\emptyset$, or, as we also say, that are not finite, that are infinite. One such cardinal number is that of the concept *finite cardinal number*; I propose to call it *Endlos* and designate it with ‘∞’. I define it thus:

$$\Vdash \not\mathfrak{p}(\emptyset\frown\nparallel\smile\mathfrak{f}) = \infty \qquad \text{(M}$$

For $\emptyset\frown\nparallel\smile\mathfrak{f}$ is the extension of the concept *finite cardinal number*. The | 150b
proposition mentioned in the heading says that the cardinal number Endlos is no finite cardinal number. We prove it, as is indicated in §84 of my *Grundlagen*, by showing that the cardinal number Endlos follows after itself in the cardinal number series, which according to (145) no finite cardinal number does. First, it is to be shown that Endlos stands in the $\mathfrak{f}$-relation to itself:

$$\text{‘}\vdash \infty\frown(\infty\frown\mathfrak{f})\text{’} \qquad (\alpha$$

| We trace this proposition back to 151

$$\text{‘}\vdash \not\mathfrak{p}\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon=\emptyset\\ \llcorner\ \varepsilon\frown(\emptyset\frown\nparallel\smile\mathfrak{f})\end{array}\right) = \infty\text{’} \qquad (\beta$$

which follows from the propositions

$$\text{‘}\vdash \dot{\varepsilon}(\emptyset\frown(\varepsilon\frown\smile\mathfrak{f}))\frown\left[\dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon=\emptyset\\ \llcorner\ \varepsilon\frown(\emptyset\frown\nparallel\smile\mathfrak{f})\end{array}\right)\frown\rangle\mathfrak{f}\right]\text{’} \qquad (\gamma$$

and

$$\text{‘}\vdash \dot{\varepsilon}\left(\begin{array}{l}\top\ \varepsilon=\emptyset\\ \llcorner\ \varepsilon\frown(\emptyset\frown\nparallel\smile\mathfrak{f})\end{array}\right)\frown(\dot{\varepsilon}(\emptyset\frown(\varepsilon\frown\smile\mathfrak{f}))\frown\rangle\nparallel\mathfrak{f})\text{’} \qquad (\delta$$

| According to (11), in order to derive (γ) we have to show 151a

$$
\text{‘}\begin{array}{l}
\vdash\!\!\top\; d\frown\dot{\varepsilon}(\emptyset\frown(\varepsilon\frown\smile\mathfrak{f})) \\
\quad\smile^{\mathfrak{a}}\!\top\; \mathfrak{a}\frown\dot{\varepsilon}\left(\begin{array}{l}\top\; \varepsilon=\emptyset\\ \llcorner\; \varepsilon\frown(\emptyset\frown\ddagger\smile\mathfrak{f})\end{array}\right)\\
\qquad\llcorner\; d\frown(\mathfrak{a}\frown\mathfrak{f})
\end{array}\text{’}, \qquad (\varepsilon
$$

which is easily traced back to the proposition

$$
\text{‘}\begin{array}{l}
\vdash\!\top\!\top\; \emptyset\frown(d\frown\smile\mathfrak{f})\\
\quad\llcorner\; d\frown(a\frown\mathfrak{f})\\
\llcorner\!\top\; a=\emptyset\\
\quad\llcorner\; a\frown(\emptyset\frown\ddagger\smile\mathfrak{f})
\end{array}\text{’} \qquad (\zeta
$$

which breaks down into the propositions

$$
\text{‘}\begin{array}{l}
\vdash\!\top\!\top\; a=\emptyset\\
\quad\llcorner\; \emptyset\frown(d\frown\smile\mathfrak{f})\\
\llcorner\; d\frown(a\frown\mathfrak{f})
\end{array}\text{’} \qquad (\eta
$$

and (137).

§123. *Construction*

[proof to be added]

152

§124. *Analysis*

Instead of proving proposition (δ) of §122, we first prove the following:[a]

$$
\text{‘}\begin{array}{l}
\vdash\!\top\; \dot{\varepsilon}\left(\begin{array}{l}\top\; \varepsilon=c\\ \llcorner\; \varepsilon\frown(c\frown\ddagger\smile q)\end{array}\right)\frown(\dot{\varepsilon}(c\frown(\varepsilon\frown\smile q))\frown\rangle\ddagger q)\\
\llcorner\; \mathrm{I}\ddagger q
\end{array}\text{’},
$$

| For this we require the proposition 152a

$$
\text{‘}\begin{array}{l}
\vdash\!\top\; d\frown\dot{\varepsilon}\left(\begin{array}{l}\top\; \varepsilon=c\\ \llcorner\; \varepsilon\frown(c\frown\ddagger\smile q)\end{array}\right)\\
\quad\smile^{\mathfrak{a}}\!\top\; \mathfrak{a}\frown\dot{\varepsilon}(c\frown(\varepsilon\frown\smile q))\\
\qquad\llcorner\; d\frown(\mathfrak{a}\frown\ddagger q)
\end{array}\text{’},
$$

which is to be traced back to the proposition

$$
\text{‘}\begin{array}{l}
\vdash\!\top\; d=c\\
\quad\llcorner\; c\frown(d\frown\smile q)\\
\quad\smile^{\mathfrak{e}}\!\top\; c\frown(\mathfrak{e}\frown\smile q)\\
\qquad\llcorner\; \mathfrak{e}\frown(d\frown q)
\end{array}\text{’},
$$

This follows straightforwardly from (142).

---

[a] *Translator's Note*: In the original, the subcomponent ‘— $\mathrm{I}\ddagger q$’ is missing; compare Frege's *corrigenda* to volume 1.

| **§125.** *Construction* 152b

[proof to be added]

154

b) Proof of the proposition

$$`\vdash \begin{array}{l} \infty = \not{\mathfrak{p}}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon\frown v\\ \neg\,\varepsilon\frown u\end{array}\right) \\ \infty = \not{\mathfrak{p}}u \\ \not{0}\frown(\not{\mathfrak{p}}v\frown\smile\mathfrak{f}) \end{array} ,$$

| **§126.** *Analysis* 154a

We now prove the proposition

> "If Endlos is the cardinal number of a concept and if the cardinal number of another concept is finite, then Endlos is the cardinal number of the concept *falling under either the first or the second concept.*"

using (144), by taking

$$`\smile^{\mathfrak{v}} \begin{array}{l} \infty = \not{\mathfrak{p}}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon\frown \mathfrak{v}\\ \neg\,\varepsilon\frown u\end{array}\right) \\ \xi = \not{\mathfrak{p}}\mathfrak{v} \end{array} ,$$

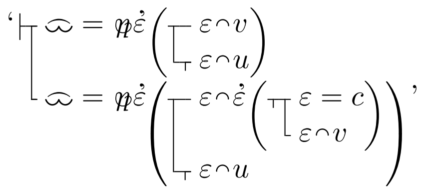
in place of the function-marker '$F(\xi)$', and first have to derive the proposition

$$`\vdash \begin{array}{l} \infty = \not{\mathfrak{p}}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon\frown v\\ \neg\,\varepsilon\frown u\end{array}\right) \\ a = \not{\mathfrak{p}}v \\ d\frown(a\frown\mathfrak{f}) \\ \smile^{\mathfrak{v}} \begin{array}{l} \infty = \not{\mathfrak{p}}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon\frown \mathfrak{v}\\ \neg\,\varepsilon\frown u\end{array}\right) \\ d = \not{\mathfrak{p}}\mathfrak{v} \end{array} \end{array} , \qquad (\alpha$$

| According to (II a) we have 154b

$$`\vdash \begin{array}{l} \infty = \not{\mathfrak{p}}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon\frown\dot{\varepsilon}\left(\begin{array}{l}\neg\,\varepsilon = c\\ \varepsilon\frown v\end{array}\right)\\ \neg\,\varepsilon\frown u\end{array}\right) \\ d = \not{\mathfrak{p}}\dot{\varepsilon}\left(\begin{array}{l}\neg\,\varepsilon = c\\ \varepsilon\frown v\end{array}\right) \\ \smile^{\mathfrak{v}} \begin{array}{l} \infty = \not{\mathfrak{p}}\dot{\varepsilon}\left(\begin{array}{l}\varepsilon\frown \mathfrak{v}\\ \neg\,\varepsilon\frown u\end{array}\right) \\ d = \not{\mathfrak{p}}\mathfrak{v} \end{array} \end{array} ,$$

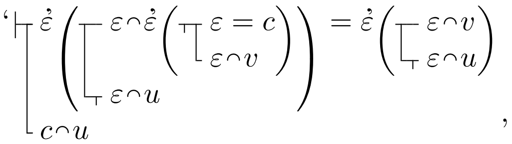
To this we can now apply (159). In order to get the desired supercomponent, we must prove the proposition



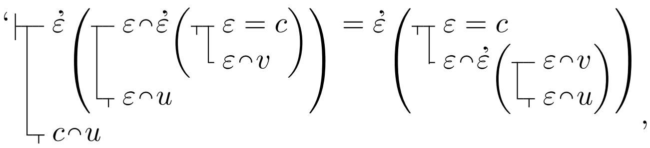
(β

To this end we distinguish the cases where c falls under the u-concept and its contrary. Thus, we have the propositions

155

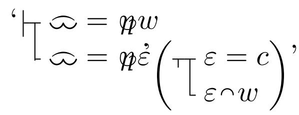


(γ



(δ

| In the second case we also require the proposition 155a



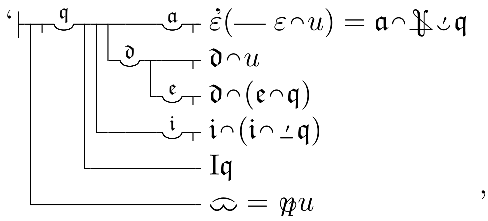
which straightforwardly follows from (165) and (69).

§127. *Construction*

[proof to be added]

160

c) Proof of the proposition



| **§128.** *Analysis* 160a

The proposition now to be proven can be expressed in words thus:

> "If Endlos is the cardinal number of a concept, then the objects falling under it can be ordered in a non-branching series which starts with a specific object and, without returning into itself, proceeds endlessly."

If $\backsim$ is the cardinal number of the u-concepts, then there must be a relation | 160b
which maps the concept *finite cardinal number* into the u-concept and whose converse maps the latter into the former. Let the p-relation be of this kind; we now ask whether then the $(\text{⅄}p \smile \mathfrak{f} \smile p)$-relation satisfies our requirements as series-forming if we take as initial member that to which $\mathfrak{0}$ stands in the p-relation. Using (17), (18) and (71), we easily prove the single-valuedness of our series-forming relation. That the series proceeds without end we can derive from (156) and (8).

| **§129.** *Construction* 161a

[proof to be added]

162a

§130. *Analysis*

It is not provable that no object follows after itself in the $(\text{⅄}p \smile \mathfrak{f} \smile p)$-series but only that, provided that the u-concept can be mapped by means of the $\text{⅄}p$-relation into the concept *finite cardinal number*, no object falling under the u-concept follows after itself in this series. For now, we make do with such a series in order then to define another one, using our $(\text{⅄}p \smile \mathfrak{f} \smile p)$-relation, which will share the here relevant properties with the former but will have, in addition, the property that none of its ob|jects follows after 162b
itself.

We prove the proposition

$$‘\vdash\!\overset{\mathfrak{i}}{\smile}\!\!\begin{array}{l} \mathfrak{i} \frown u \\ \mathfrak{i} \frown (\mathfrak{i} \frown \dot{\bot}(\text{⅄}p \smile \mathfrak{f} \smile p)) \\ u \frown (\mathfrak{0} \frown \text{⅄} \dot{\smile} \mathfrak{f} \frown \rangle \text{⅄}p) \end{array}’$$

from the propositions

$$‘\vdash\!\!\begin{array}{l} x \frown (y \frown (\text{⅄}p \smile \dot{\bot}q \smile p)) \\ \mathrm{I}\text{⅄}p \\ x \frown (y \frown \dot{\bot}(\text{⅄}p \smile q \smile p)) \end{array}’ \qquad (\alpha$$

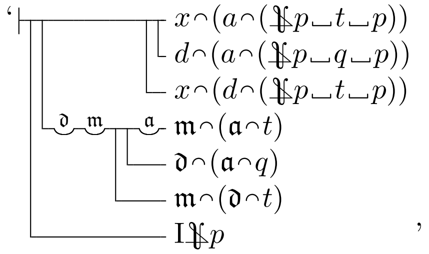
and

$$‘\vdash\!\!\begin{array}{l} x \frown u \\ x \frown (x \frown (\text{⅄}p \smile \dot{\bot}\mathfrak{f} \smile p)) \\ \mathrm{I}\text{⅄}p \\ u \frown (\mathfrak{0} \frown \text{⅄} \dot{\smile} \mathfrak{f} \frown \rangle \text{⅄}p) \end{array}’ \qquad (\beta$$

The former we prove with (123), and for this we require the proposition

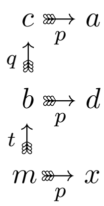
$$‘\vdash\!\!\begin{array}{l} x \frown (a \frown (\text{⅄}p \smile \dot{\bot}q \smile p)) \\ d \frown (a \frown (\text{⅄}p \smile q \smile p)) \\ x \frown (d \frown (\text{⅄}p \smile \dot{\bot}q \smile p)) \\ \mathrm{I}\text{⅄}p \end{array}’ \qquad (\gamma$$

which can be derived from the more general proposition[a]



, (δ

For the proof of (δ) we will draw on objects, say, m, b, c, which stand to x, d, a in the p-relation. That there are such objects is to be shown by means of (15). Here, b occurs twice: first, by m's standing to it in the t-relation, and second, by standing in the q-relation to c. The following diagram may help make things more surveyable: 163a



From the single-valuedness of the converse of the p-relation we must conclude that there is only a single object of this kind that can come into consideration.

§131. *Construction*

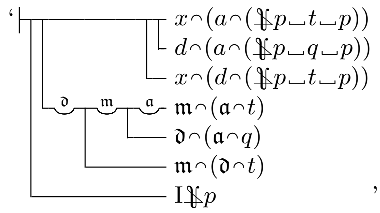
[proof to be added]

165b

§132. *Analysis*

We will now prove the proposition (β) of §130 by inferring from the single-valuedness of the ⧗p-relation that there is only one object which stands in this relation to x; whereas if x were to stand to itself in the

[a] *Translators' Note*: Error in the following formula, discovered by William Stirton; the formula in the original looks like this:



,

— this leaves the '𝔪' in the second lowest subcomponent unbound. Compare (π) on p. 164, which shows the correct formula.

($\mathfrak{p}p \smile \acute{-}\ \mathfrak{f} \smile p$)-relation, there would have to be, according to (15), at least one object which followed after itself in the cardinal number series and which thus, according to (145), could not be a finite cardinal number. From this it follows, in accordance with (8), that x could not fall under the u-concept if the u-concept is mapped by the $\mathfrak{p}p$-relation into the concept *finite cardinal number*.

§133. *Construction*

[proof to be added]

§134. *Analysis* 167a

It now only remains to show that all the members of our series fall under the u-concept and conversely that all objects falling under the u-concept are members of our series. These are the two propositions

$$`\vdash \begin{array}{l} y \frown u \\ x \frown (y \frown \acute{\smile}(\mathfrak{p}p \smile q \smile p)) \\ m \frown (x \frown p) \\ m \frown \mathfrak{p}\acute{\smile} q \frown (u \frown \rangle p) \\ \mathrm{I}\mathfrak{p}p \end{array} \text{'}, \qquad (\alpha$$

and

$$`\vdash \begin{array}{l} x \frown (y \frown \acute{\smile}(\mathfrak{p}p \smile q \smile p)) \\ y \frown u \\ m \frown (x \frown p) \\ m \frown \mathfrak{p}\acute{\smile} q \frown (u \frown \rangle p) \\ u \frown (m \frown \mathfrak{p}\acute{\smile} q \frown \rangle \mathfrak{p}p) \end{array} \text{'}, \qquad (\beta$$

where the q-series starting with m is taken for the cardinal number series more generally. We prove (α) from the propositions

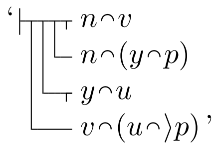
$$`\vdash \begin{array}{l} x \frown (y \frown (\mathfrak{p}p \smile \acute{\smile} q \smile p)) \\ m \frown (x \frown p) \\ \mathrm{I}\mathfrak{p}p \\ x \frown (y \frown \acute{\smile}(\mathfrak{p}p \smile q \smile p)) \end{array} \text{'} \qquad (\gamma$$

and

$$`\vdash \begin{array}{l} y \frown u \\ m \frown (x \frown p) \\ x \frown (y \frown (\mathfrak{p}p \smile \acute{\smile} q \smile p)) \\ m \frown \mathfrak{p}\acute{\smile} q \frown (u \frown \rangle p) \\ \mathrm{I}\mathfrak{p}p \end{array} \text{'}, \qquad (\delta$$

of which (γ) is easily deduced similar to (177). In order to prove (δ) we infer from the single-valuedness of the $\mathfrak{p}p$-relation that there is only one object

which stands to x in the $\mathfrak{I}p$-relation, and we infer from x's standing to y in the $(\mathfrak{I}p \llcorner \dot{\bot}\, \mathfrak{f} \llcorner p)$-relation | that there is such an object that belongs to a 167b
q-series ending with an object n which stands to y in the p-relation. Thus if an object m stands in the $\mathfrak{I}p$-relation to x, it will also belong to the q-series ending with n. We will further prove the proposition



and arrive at our aim by taking the $(m \frown \mathfrak{I} \dot{\smile} q)$-concept here as the v-concept.

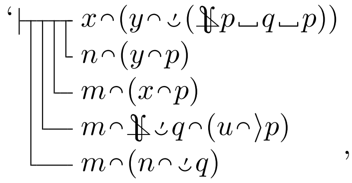
§135. *Construction*

[proof to be added]

169b

§136. *Analysis*

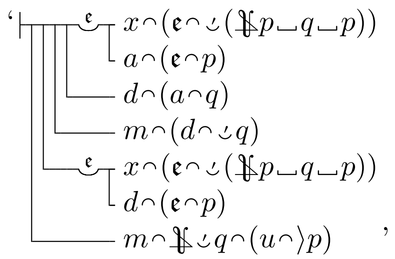
We now have to prove the proposition (β) of §134. From the u-concept's being mapped into the $(m \frown \mathfrak{I} \dot{\smile} q)$-concept by means of the $\mathfrak{I}p$-relation, and y's falling under the u-concept, we can infer that there is an object (n) to which y stands in the $\mathfrak{I}p$-relation and which falls under the $(m \frown \mathfrak{I} \dot{\smile} q)$-concept, i.e., it belongs to a q-series beginning with m. We now prove the proposition



(α

169b

with (152). We require for this the proposition



(β

From the $(m \frown \mathfrak{I} \dot{\smile} q)$-concept's being mapped into the u-concept by the p-relation, we infer that there is an object (e) to which d stands in the p-relation if d belongs to the q-series starting with m. From this and the proposition

$$‘\begin{array}{l} \vdash\ x\frown(c\frown\smallsmile(\not\setminus p\smile q\smile p)) \\ \quad a\frown(c\frown p) \\ \quad d\frown(a\frown q) \\ \quad d\frown(e\frown p) \\ \quad x\frown(e\frown\smallsmile(\not\setminus p\smile q\smile p))’ \end{array} \qquad (\gamma$$

(β) easily follows.

§137. *Construction*

[proof to be added]

171b

§138. *Analysis*

Now, as was promised in §130, we define a relation of the kind that no object follows after itself in its series and which otherwise coincides with the ($\not\setminus p\smile \mathfrak{f} \smile p$)-relation in respect of the properties that matter to us.

$$\Vdash\ \dot{\alpha}\dot{\varepsilon}\left(\begin{array}{l} \top\ \varepsilon\frown(\alpha\frown q) \\ \quad \alpha\frown u \end{array}\right) = u\frown q \qquad (\mathrm{N}$$

We now show that the ($u\frown q$)-relation has these properties if the q-relation has them, and that no object follows after itself in the ($u\frown q$)-relation if no object that falls under the u-concept follows after itself in the q-series. We will first prove the propositions

$$‘\begin{array}{l} \vdash\ \mathrm{I}(u\frown q) \\ \quad \mathrm{I}q \end{array}’ \quad (\alpha \qquad ‘\begin{array}{l} \vdash\!\overset{\mathfrak{i}}{\smile}\ \mathfrak{i}\frown(\mathfrak{i}\frown\smallsmile(u\frown q)) \\ \quad \overset{\mathfrak{i}}{\smile}\ \mathfrak{i}\frown u \\ \qquad \mathfrak{i}\frown(\mathfrak{i}\frown\smallsmile q) \end{array}’ \quad (\beta$$

| The first does not present any difficulty; (β) can be broken up into the propositions 172a

$$‘\begin{array}{l} \vdash\ y\frown u \\ \quad x\frown(y\frown\smallsmile(u\frown q))’ \end{array} \qquad (\gamma$$

$$‘\begin{array}{l} \vdash\ x\frown(y\frown\smallsmile q) \\ \quad x\frown(y\frown\smallsmile(u\frown q))’ \end{array} \qquad (\delta$$

§139. *Construction*

[proof to be added]

173a

§140. *Analysis*

We now have to prove that under these assumptions the ($u\frown(\not\setminus p\smile\mathfrak{f}\smile p)$)-series proceeds endlessly. This turns on the proposition

$$‘\begin{array}{l} \vdash\ d\frown(a\frown(\not\setminus p\smile \mathfrak{f} \smile p)) \\ \quad \overset{\mathfrak{e}}{\smile}\ d\frown(\mathfrak{e}\frown(u\frown(\not\setminus p\smile \mathfrak{f} \smile p))) \\ \quad d\frown u \\ \quad \vartheta\frown(x\frown p) \\ \quad \vartheta\frown\not\setminus\smallsmile\mathfrak{f}\frown(u\frown\rangle p) \\ \quad u\frown(\vartheta\frown\not\setminus\smallsmile\mathfrak{f}\frown\rangle\not\setminus p) \end{array}’ \qquad (\alpha$$

| which is to be proven using the proposition 173b

$$
\text{‘}\vdash\begin{array}{l} d\frown(a\frown(u\bumpeq q)) \\ \llcorner\, d\frown(a\frown q) \\ \llcorner\, a\frown u \end{array}\text{’,}
$$

which follows from (N).

§141. *Construction*

[proof to be added]

§142. *Analysis* 174

It now remains to prove the proposition

$$
\text{‘}\vdash\begin{array}{l} (\text{—}\, y\frown u) = x\frown(y\frown\dot{\smile}(u\bumpeq(\mathfrak{J}p\sqcup\mathfrak{f}\sqcup p))) \\ \llcorner\, \emptyset\frown(x\frown p) \\ \llcorner\, \emptyset\frown\mathfrak{J}\dot{\smile}\mathfrak{f}\frown(u\frown\rangle p) \\ \llcorner\, u\frown(\emptyset\frown\mathfrak{J}\dot{\smile}\mathfrak{f}\frown\rangle\mathfrak{J}p) \end{array}\text{’,} \qquad (\alpha
$$

| Using the propositions (181) and (186) this can be traced back to the propositions 174a

$$
\text{‘}\vdash\begin{array}{l} x\frown(y\frown\dot{\smile}q) \\ \llcorner\, x\frown(y\frown\dot{\smile}(u\bumpeq q)) \end{array}\text{’,} \qquad (\beta
$$

and

$$
\text{‘}\vdash\begin{array}{l} x\frown(y\frown\dot{\smile}(u\bumpeq(\mathfrak{J}p\sqcup\mathfrak{f}\sqcup p))) \\ \llcorner\, \emptyset\frown(x\frown p) \\ \llcorner\, \emptyset\frown\mathfrak{J}\dot{\smile}\mathfrak{f}\frown(u\frown\rangle p) \\ \llcorner\, u\frown(\emptyset\frown\mathfrak{J}\dot{\smile}\mathfrak{f}\frown\rangle\mathfrak{J}p) \\ \llcorner\, x\frown(y\frown\dot{\smile}(\mathfrak{J}p\sqcup\mathfrak{f}\sqcup p)) \end{array}\text{’,} \qquad (\gamma
$$

| Of these (β) can be derived by means of (194), whereas (γ) can be proven using (154). For this we require the proposition 174b

$$
\text{‘}\vdash\begin{array}{l} x\frown(a\frown\dot{\smile}(u\bumpeq(\mathfrak{J}p\sqcup\mathfrak{f}\sqcup p))) \\ \llcorner\, d\frown(a\frown(\mathfrak{J}p\sqcup\mathfrak{f}\sqcup p)) \\ \llcorner\, x\frown(d\frown\dot{\smile}(\mathfrak{J}p\sqcup\mathfrak{f}\sqcup p)) \\ \llcorner\, x\frown(d\frown\dot{\smile}(u\bumpeq(\mathfrak{J}p\sqcup\mathfrak{f}\sqcup p))) \\ \llcorner\, \emptyset\frown(x\frown p) \\ \llcorner\, \emptyset\frown\mathfrak{J}\dot{\smile}\mathfrak{f}\frown(u\frown\rangle p) \\ \llcorner\, u\frown(\emptyset\frown\mathfrak{J}\dot{\smile}\mathfrak{f}\frown\rangle\mathfrak{J}p) \end{array}\text{’,} \qquad (\delta
$$

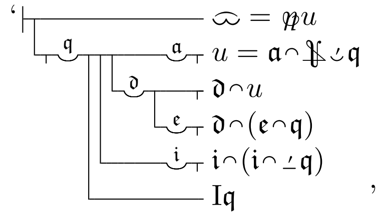
which follows from (198) and (137).

| We then easily arrive at the end of our section (c). 175a

§143. *Construction*

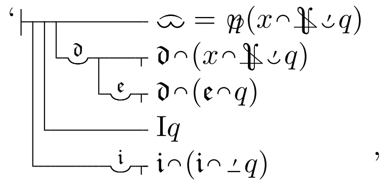
[proof to be added]

d) Proof of the proposition



§144. *Analysis* 179a

We now prove the converse of proposition (207), namely that Endlos is the cardinal number which belongs to a concept, if the objects falling under that concept can be ordered into a series that starts with a certain object and proceeds endlessly without looping back into itself and without branching. It turns on showing that Endlos is the cardinal number that belongs to the concept *member of such a series*; in signs



(α

We use proposition (32) for this and need to prove that there is a relation which maps the cardinal number series into the q-series starting with x and whose converse maps the latter into the former. It suggests itself to correlate 0 with x, 1 with the member immediately following x in the q-series and, in this manner, each immediately following cardinal number to the immediately following member of the q-series. We always pair one member | 179b of the cardinal number series with one member of the q-series and form a series out of these pairs. The series-forming relation is determined thus: one pair stands in it to a second if the first member of the first pair stands in the $\mathfrak{f}$-relation to the first member of the second pair and the second member of the first pair stands in the q-relation to the second member of the second pair. If the pair $(n; y)$ belongs to the series that starts with the pair $(0; x)$, then n stands to y in the mapping relation that is to be demonstrated. We now define the pair in this way:

$$\Vdash \dot{\varepsilon}(o\frown(a\frown\varepsilon)) = o; a \qquad (\Xi$$

The semicolon is here a two-sided function-sign. The expression

'$\Pi\frown(\Gamma; \Delta)$'

is thus co-referential with

$$\text{‘}\Gamma\frown(\Delta\frown\Pi)\text{’}$$

provided ‘Γ’, ‘Δ’ and ‘Π’ refer to objects. For the extension of the relation given in the manner described above by, as I say, *coupling* the p-relation and the q-relation, I will introduce a simple sign by defining: 179

$$\Vdash \dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l} \top\!\overset{\mathfrak{a}}{\smile}\!\overset{\mathfrak{o}}{\smile}\!\!\!-\!\overset{\mathfrak{d}}{\smile}\!\!\!-\!\overset{\mathfrak{c}}{\smile}\!\!\!-\!\top\; \mathfrak{c}\frown(\mathfrak{o}\frown p) \\ \qquad\qquad\quad\;\; \llcorner\; \varepsilon = \mathfrak{c};\mathfrak{d} \\ \qquad\qquad \llcorner\; \mathfrak{d}\frown(\mathfrak{a}\frown q) \\ \qquad \llcorner\; \alpha = \mathfrak{o};\mathfrak{a} \end{array}\right] = p\simeq q \qquad (\mathrm{O}$$

Accordingly,

$$\text{‘}\emptyset; x\frown(\xi;\zeta\frown\acute{\smile}(\mathfrak{f}\simeq q))\text{’}$$

indicates our mapping relation and

$$\text{‘}\dot{\alpha}\dot{\varepsilon}(\emptyset; x\frown(\varepsilon;\alpha\frown\acute{\smile}(\mathfrak{f}\simeq q)))\text{’}$$

its extension. We now define

$$\Vdash \dot{\alpha}\dot{\varepsilon}(A\frown(\varepsilon;\alpha\frown\acute{\smile}t)) = A\nwarrow t \qquad (\Pi$$

| The propositions 180

$$\text{‘}\begin{array}{l} \vdash\!\!\top\!\!\top\!\!-\!-\; \emptyset\frown\not\mathfrak{F}\acute{\smile}\,\mathfrak{f}\frown(x\frown\not\mathfrak{F}\acute{\smile}q\frown\rangle(\emptyset;x\nwarrow(\mathfrak{f}\simeq q))) \\ \quad\;\llcorner\overset{\mathfrak{d}}{\smile}\!\!\top\!\!-\!\top\; \mathfrak{d}\frown(x\frown\not\mathfrak{F}\acute{\smile}q) \\ \qquad\quad\llcorner\overset{\mathfrak{e}}{\smile}\!\top\; \mathfrak{d}\frown(\mathfrak{e}\frown q) \\ \;\;\llcorner\!-\!-\!-\!-\; \mathrm{I}q \end{array} \text{’} \qquad (\beta$$

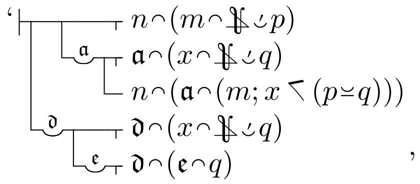
and

$$\text{‘}\begin{array}{l} \vdash\!\!\top\!\!\top\!\!-\!-\; x\frown\not\mathfrak{F}\acute{\smile}q\frown(\emptyset\frown\not\mathfrak{F}\acute{\smile}\,\mathfrak{f}\frown\rangle\not\mathfrak{F}(\emptyset;x\nwarrow(\mathfrak{f}\simeq q))) \\ \quad\;\llcorner\overset{\mathfrak{i}}{\smile}\!\top\; \mathfrak{i}\frown(\mathfrak{i}\frown\acute{-}q) \\ \;\;\llcorner\!-\!-\!-\; \mathrm{I}q \end{array} \text{’} \qquad (\gamma$$

then have to be proven. Instead of (β) we first prove the somewhat more general proposition

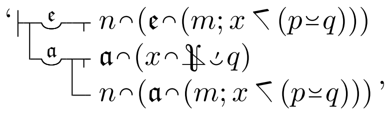
$$\text{‘}\begin{array}{l} \vdash\!\!\top\!\!\top\!\!\top\!\!\top\!\!-\!-\; m\frown\not\mathfrak{F}\acute{\smile}p\frown(x\frown\not\mathfrak{F}\acute{\smile}q\frown\rangle(m;x\nwarrow(p\simeq q))) \\ \qquad\llcorner\overset{\mathfrak{d}}{\smile}\!\!\top\!\!-\!\top\; \mathfrak{d}\frown(x\frown\not\mathfrak{F}\acute{\smile}q) \\ \qquad\qquad\llcorner\overset{\mathfrak{e}}{\smile}\!\top\; \mathfrak{d}\frown(\mathfrak{e}\frown q) \\ \quad\;\;\llcorner\!-\!-\!\overset{\mathfrak{i}}{\smile}\!\!\top\!\top\; \mathfrak{i}\frown(\mathfrak{i}\frown\acute{-}p) \\ \qquad\qquad\quad\llcorner\; m\frown(\mathfrak{i}\frown\acute{\smile}p) \\ \quad\llcorner\!-\!-\!-\!-\; \mathrm{I}p \\ \;\;\llcorner\!-\!-\!-\!-\; \mathrm{I}q \end{array} \text{’} \qquad (\delta$$

| which we can then also use for the proof of (γ). We use (11) and thus have to deduce the proposition 180a



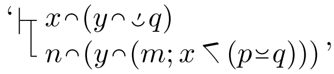
(ε

The proposition



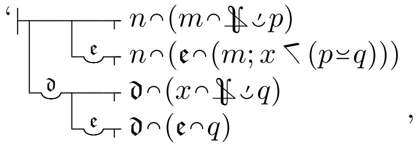
(ζ

is easily deduced from the proposition



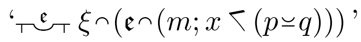
(η

and with the latter we can trace (ε) back to

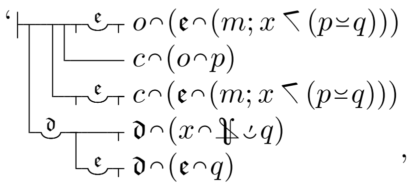


(ϑ

| We prove this proposition using (144), by putting the function-marker 180b



for ‘$F(\xi)$’. We then require the proposition



(ι

In order to aid comprehension, I give the following diagram of the p-series and the q-series placed next to each other:

p-series	q-series
m	x
⋮	⋮
c	d
o	a

It is to be shown:

“If the q-series starting with x proceeds without end, and if there is an object (d) which together with c forms a pair which belongs to the $p{\asymp}q$-series starting with the pair $m;x$, then there is also an object (a) which forms such a pair with o provided c stands to o in the p-relation.” 181a

We first prove the proposition

$$
\begin{array}{l}
\text{‘}\vdash\; o\frown(a\frown(A\nwarrow(p\stackrel{\smile}{=}q))) \\
\quad d\frown(a\frown q) \\
\quad c\frown(o\frown p) \\
\quad c\frown(d\frown(A\nwarrow(p\stackrel{\smile}{=}q)))\text{’}
\end{array}
\quad [1] \qquad (\kappa
$$

for which, because of definition (Π), we can write 181b

$$
\begin{array}{l}
\text{‘}\vdash\; A\frown(o;a\frown\overset{\prime}{\smile}(p\stackrel{\smile}{=}q)) \\
\quad d\frown(a\frown q) \\
\quad c\frown(o\frown p) \\
\quad A\frown(c;d\frown\overset{\prime}{\smile}(p\stackrel{\smile}{=}q))\text{’}
\end{array}
\qquad (\lambda
$$

This can be proven straightforwardly by means of the proposition

$$
\begin{array}{l}
\text{‘}\vdash\; c;d\frown(o;a\frown(p\stackrel{\smile}{=}q)) \\
\quad d\frown(a\frown q) \\
\quad c\frown(o\frown p)
\end{array}
\text{’} \qquad (\mu
$$

which follows from definition (O).

§145. *Construction*

[proof to be added]

§146. *Analysis* 183a

We now have to eliminate the subcomponent in (212)

$$\text{‘}—\; x\frown(d\frown\overset{\prime}{\smile}q)\text{’}$$

For this, we employ the proposition (η) of §144, which follows from

$$
\begin{array}{l}
\text{‘}\vdash\; x\frown(d\frown\overset{\prime}{\smile}q) \\
\quad m;x\frown(c;d\frown\overset{\prime}{\smile}(p\stackrel{\smile}{=}q))\text{’}
\end{array}
\qquad (\alpha
$$

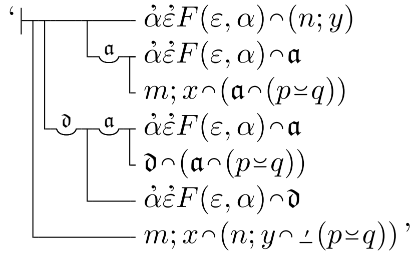
| We will first prove the proposition 183b

$$
\begin{array}{l}
\text{‘}\vdash\; m\frown(c\frown\overset{\prime}{\_}p) \\
\quad x\frown(d\frown\overset{\prime}{\_}q) \\
\quad m;x\frown(c;d\frown\overset{\prime}{\_}(p\stackrel{\smile}{=}q))\text{’}
\end{array}
\qquad (\beta
$$

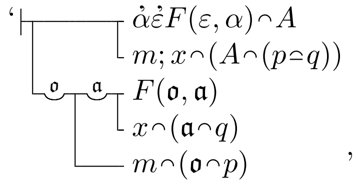
which we will also use on other occasions. For this we need the proposition

$$
\begin{array}{l}
\text{‘}\vdash\; F(n,y) \\
\quad \smile^{\mathfrak{o}}\smile^{\mathfrak{a}}\; F(\mathfrak{o},\mathfrak{a}) \\
\qquad x\frown(\mathfrak{a}\frown q) \\
\qquad m\frown(\mathfrak{o}\frown p) \\
\quad \smile^{\mathfrak{c}}\smile^{\mathfrak{d}}\smile^{\mathfrak{o}}\smile^{\mathfrak{a}}\; F(\mathfrak{o},\mathfrak{a}) \\
\qquad \mathfrak{d}\frown(\mathfrak{a}\frown q) \\
\qquad \mathfrak{c}\frown(\mathfrak{o}\frown p) \\
\qquad F(\mathfrak{c},\mathfrak{d}) \\
\quad m;x\frown(n;y\frown\overset{\prime}{\_}(p\stackrel{\smile}{=}q))\text{’}
\end{array}
\qquad (\gamma
$$

| which is similar to proposition (123) and can be proven by means of it. To this end we write (123) in the form 183a

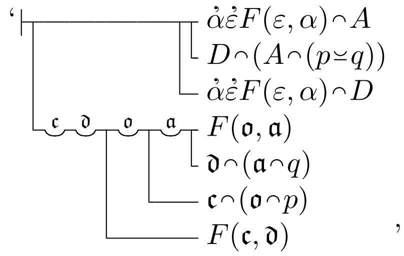


First, we have to prove the proposition



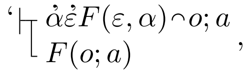
(δ

from which further the proposition

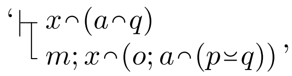


(ε

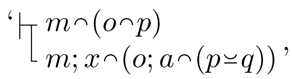
| which we also need, follows. From the propositions 183b



(ζ

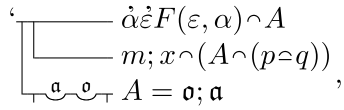


(η



(ϑ

we can easily arrive at a proposition which differs from (δ) only in that ‘$o; a$’ stands in the place of ‘A’. The supercomponent which consists of the first two lines may then be replaced by



In order to eliminate the subcomponent ‘$\smile^{\mathfrak{a}}\smile^{\mathfrak{o}}$ $A = \mathfrak{o}; \mathfrak{a}$’, we use the proposition

[1] Here “A” is written for “$m; x$”.

$$`\vdash\!\!\top\!\overset{\mathfrak{a}}{\smile}\!\overset{\mathfrak{o}}{\smile}\!\top\; A = \mathfrak{o};\mathfrak{a}$$
$$\llcorner\; D \frown (A \frown (p \veebar q))\,' \qquad (\iota$$

which follows from (O).

§147. *Construction* 184

[proof to be added]

§148. *Analysis* 184a

We will use (213) in order to prove the propositions (η) and (ϑ), requiring additionally the proposition

$$\begin{array}{l} `\vdash\!\top\!\top\!\top\!\top\; c \frown (e \frown p) \\ \quad\llcorner\; m; x = c; d \\ \quad\llcorner\; d \frown (i \frown q) \\ \quad\llcorner\; o; a = e; i \\ \quad\llcorner\!\top\; m \frown (o \frown p) \\ \quad\;\;\llcorner\; x \frown (a \frown q)\;\; ' \end{array} \qquad (\alpha$$

which we can prove using the proposition | 184b

$$\begin{array}{l} `\vdash\!\top\!\top\!\top\; m = c \\ \quad\llcorner\; x = d \\ \quad\llcorner\; m; x = c; d\,' \end{array}$$

The latter follows from (Ξ).

§149. *Construction*

[proof to be added]

§150. *Analysis* 189a

In order now to derive the proposition (β) of §146, we replace the function-marker ‘$F(\xi, \zeta)$’ in (231) by

$$\begin{array}{l} `\top\!\top\; m \frown (\xi \frown \underline{\smile} p) \\ \quad\llcorner\; x \frown (\zeta \frown \underline{\smile} q)\;\; ' \end{array}$$

The propositions that are needed for this are straightforwardly proven from (133) and (131).

§151. *Construction*

[proof to be added]

In order to obtain our proposition (γ) of §144, we have to replace the subcomponent in (241)

$$\text{‘}— \mathrm{I}(m; x \nwarrow (p \underline{\smile} q))\text{’}$$

by others. The train of thought here is as follows. If the pairs $(b; d)$ and $(b; a)$ belong to the $(p \underline{\smile} q)$-series starting with $(m; x)$, then either $(b; d)$ has to belong to the $(p \underline{\smile} q)$-series starting with $(b; a)$, or [$(b; a)$ has][a] to follow after $(b; d)$ in this series, insofar the $(p \underline{\smile} q)$-relation is single-valued. If either $(b; a)$ follows after $(b; d)$ or $(b; d)$ follows after $(b; a)$ in this series, then b has to follow after itself in the p-series, which would contradict our subcomponent

$$\text{‘}\overset{\mathfrak{i}}{\smile}\!\!\begin{array}{l} \neg\; \mathfrak{i} \frown (\mathfrak{i} \frown \underline{\cdot} p) \\ m \frown (\mathfrak{i} \frown \dot{\smile} p)\text{’} \end{array}$$

The only remaining possibility is that $(b; d)$ coincides with $(b; a)$. In that case d also coincides with a.

We therefore need the proposition

$$\text{‘}\vdash \begin{array}{l} r \frown (n \frown \dot{\smile} p) \\ \neg\; n \frown (r \frown \underline{\cdot} p) \\ m \frown (n \frown \dot{\smile} p) \\ \mathrm{I}p \\ m \frown (r \frown \dot{\smile} p)\text{’} \end{array} \qquad (\alpha$$

| in words: “If a first and a second objects belong to the p-series starting with 191b
a third, then the first precedes the second or belongs to the series starting with the second, provided the series-forming relation is single-valued.”

We prove the proposition by means of (144), replacing the function-marker ‘$F(\xi)$’ by ‘$\begin{array}{l} \xi \frown (n \frown \dot{\smile} p) \\ \neg\; n \frown (\xi \frown \underline{\cdot} p) \end{array}$’.

We then have to prove the proposition

$$\text{‘}\vdash \begin{array}{l} a \frown (n \frown \dot{\smile} p) \\ \neg\; n \frown (a \frown \underline{\cdot} p) \\ d \frown (a \frown p) \\ d \frown (n \frown \dot{\smile} p) \\ \neg\; n \frown (d \frown \underline{\cdot} p) \\ \mathrm{I}p \end{array}\text{’} \qquad (\beta$$

For this we need the proposition

$$\text{‘}\vdash \begin{array}{l} a \frown (n \frown \dot{\smile} p) \\ d \frown (a \frown p) \\ \mathrm{I}p \\ d \frown (n \frown \underline{\cdot} p)\text{’} \end{array} \qquad (\gamma$$

[a] *Translators’ Note*: The original misses out “$(b; a)$ has” in this sentence. We assume that this was a typo.

which we deduce from the propositions

$$\text{‘}\vdash \begin{array}{l} a \frown (n \frown \overset{\prime}{\smile} p) \\ \smile^{\mathfrak{a}}\; a \frown (\mathfrak{a} \frown \overset{\prime}{\smile} p) \\ \quad d \frown (\mathfrak{a} \frown p) \\ d \frown (n \frown \overset{\prime}{-} p)\text{’} \end{array} \qquad (\delta$$

| and 192a

$$\text{‘}\vdash \begin{array}{l} \smile^{\mathfrak{a}}\; a \frown (\mathfrak{a} \frown \overset{\prime}{\smile} p) \\ \quad d \frown (\mathfrak{a} \frown p) \\ d \frown (a \frown p) \\ \mathrm{I} p \text{’} \end{array} \qquad (\varepsilon$$

§153. *Construction*

[proof to be added]

§154. *Analysis* 194a

We now further prove the proposition

$$\text{‘}\vdash \begin{array}{l} \mathrm{I}(p \asymp q) \\ \mathrm{I} p \\ \mathrm{I} q \text{’} \end{array} \qquad (\alpha$$

| For this we need the proposition 194b

$$\text{‘}\vdash \begin{array}{l} o; a = e; i \\ o = e \\ a = i \text{’} \end{array} \qquad (\beta$$

which is to be derived from (Ξ). From this with (13) we gain the proposition

$$\text{‘}\vdash \begin{array}{l} o; a = e; i \\ c \frown (e \frown p) \\ c \frown (o \frown p) \\ \mathrm{I} p \\ d \frown (i \frown q) \\ d \frown (a \frown q) \\ \mathrm{I} q \text{’} \end{array} \qquad (\gamma$$

We introduce ‘D’ for ‘$o; a$’ and ‘A’ for ‘$e; i$’ and, after introducing German letters, apply (213).

§155. *Construction*

[proof to be added]

§156. *Analysis* 197

In (256) we have proposition (β) of §144. (γ) still remains to be proven . We will use (254) for this, by taking 'q' for 'p', '$\mathfrak{Q}$' for 'x', 'x' for 'm' and '$\mathfrak{f}$' for 'q'. The resulting subcomponent

$$\text{‘}\overset{\mathfrak{d}}{\smile}\!\!\top\; \mathfrak{d} \frown (\mathfrak{Q} \frown \rangle\!\!\!\lfloor \smile \mathfrak{f})$$
$$\quad \llcorner \overset{\mathfrak{e}}{\smile}\!\!\top\; \mathfrak{d} \frown (\mathfrak{e} \frown \mathfrak{f})\text{’},$$

may be eliminated by using (156). It remains to derive the more general proposition

$$\text{‘}\vdash x; m \mathbin{\overline{\nwarrow}} (q \succeq p) = \rangle\!\!\!\lfloor (m; x \mathbin{\overline{\nwarrow}} (p \succeq q))\text{’} \qquad (\alpha$$

where 'x' occurs instead of '$\mathfrak{Q}$' and 'p' instead of '$\mathfrak{f}$'. This proposition may be reduced to

$$\text{‘}\vdash x; m \frown (y; n \frown (q \succeq p)) = m; x \frown (n; y \frown \smile (p \succeq q))\text{’} \qquad (\beta$$

We derive (β) from

$$\text{‘}\vdash\!\!\top\; x; m \frown (y; n \frown (q \succeq p))$$
$$\quad \llcorner\; m; x \frown (n; y \frown \smile (p \succeq q))\text{’} \qquad (\gamma$$

which we prove by means of the proposition

$$\text{‘}\vdash\;\; F(n, y)$$
$$\quad \llcorner\; F(m, x)$$
$$\quad \llcorner \overset{\mathfrak{c}}{\smile}\overset{\mathfrak{d}}{\smile}\overset{\mathfrak{o}}{\smile}\overset{\mathfrak{a}}{\smile}\top\; F(\mathfrak{o}, \mathfrak{a})$$
$$\quad \llcorner\; \mathfrak{d} \frown (\mathfrak{a} \frown q)$$
$$\quad \llcorner\; \mathfrak{c} \frown (\mathfrak{o} \frown p)$$
$$\quad \llcorner\; F(\mathfrak{c}, \mathfrak{d})$$
$$\quad \llcorner\; m; x \frown (n; y \frown \smile (p \succeq q))\text{’} \qquad (\delta$$

(δ) follows from (230) and (144).

198

§157. *Construction*

[proof to be added]

K. Proof of the proposition

$$\text{‘}\vdash\!\!\top\; \mathfrak{Q} \frown (\not{p}u \frown \smile \mathfrak{f})$$
$$\quad \llcorner \overset{\mathfrak{A}}{\smile}\overset{\mathfrak{q}}{\smile}\top\; u = \mathfrak{A} \mathbin{\underline{\measuredangle}} \mathfrak{q}\text{’}$$

a) Proof of the proposition

$$\text{‘}\vdash\!\!\top\; \not{p}(x; y \mathbin{\underline{\measuredangle}} q) = \not{p}(\mathbb{1}; n \mathbin{\underline{\measuredangle}} \mathfrak{f})$$
$$\quad \llcorner\top\; y \frown (y \frown \underline{\smile}\, q)$$
$$\quad \llcorner\; \mathrm{I}q$$
$$\quad \llcorner\; x; \mathbb{1} \frown (y; n \frown \smile (q \succeq \mathfrak{f}))\text{’},$$

| **§158.** *Analysis* 201a

For finite cardinals we can prove a proposition similar to the last, namely that the cardinal number of a concept is finite if the objects falling under it can be ordered into a *simple* (non-branching, not looping back into itself) series starting with a certain object and ending with a certain object. For this, we require an abbreviation which we will introduce as follows:

$$\Vdash \grave{\varepsilon}\left[\begin{array}{l} \smile^{\mathfrak{n}}\smile^{\mathfrak{m}}\ \varepsilon \frown (\mathfrak{n}\frown \underset{\smile}{\frown} q) \\ \quad \mathfrak{m} \frown (\varepsilon \frown \underset{\smile}{\frown} q) \\ \quad \mathfrak{n} \frown (\mathfrak{n} \frown \overset{\frown}{-} q) \\ \quad A = \mathfrak{m}; \mathfrak{n} \\ \quad \mathrm{I}q \end{array}\right] = A \underset{-}{\lesssim} q \qquad (\mathrm{P}$$

201b

If Γ, Δ, Θ are objects and Υ the extension of a relation, then

$$\text{‘}\Gamma \frown (\Delta; \Theta) \underset{-}{\lesssim} \Upsilon\text{’}$$

says that Γ belongs to the Υ-series starting with Δ and ending with Θ, where the Υ-relation is single-valued and Θ does not follow itself in the Υ-series. We express this in words for short as follows: "*Γ belongs to the Υ-series running from Δ to Θ*". With the notation thus explained our proposition assumes the form displayed in the main heading. For

$$\Delta; \Theta \underset{-}{\lesssim} \Upsilon$$

is the extension of the concept | *belonging to the Υ-series running from Δ to Θ*. 202a

We first prove the proposition

$$\text{‘}\vdash \begin{array}{l} \mathfrak{0} \frown (\mathfrak{p}(x; y \underset{-}{\lesssim} q) \frown \underset{\smile}{\frown} \mathfrak{f}) \\ y \frown (y \frown \overset{\frown}{-} q) \\ \mathrm{I}q \\ x; \mathfrak{1} \frown (y; n \frown \underset{\smile}{\frown} (q \simeq \mathfrak{f}))\text{’} \end{array} \qquad (\alpha$$

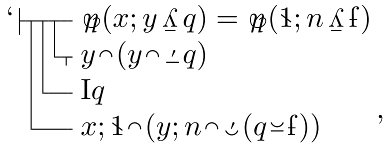
from which the subcomponents then are to be removed. (α) is obtained from (234) in the form

$$\text{‘}\vdash \begin{array}{l} \mathfrak{1} \frown (n \frown \underset{\smile}{\frown} \mathfrak{f}) \\ x; \mathfrak{1} \frown (y; n \frown \underset{\smile}{\frown} (q \simeq \mathfrak{f}))\text{’} \end{array}$$

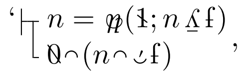
and the proposition

$$\text{‘}\vdash \begin{array}{l} \mathfrak{p}(x; y \underset{-}{\lesssim} q) = n \\ y \frown (y \frown \overset{\frown}{-} q) \\ \mathrm{I}q \\ x; \mathfrak{1} \frown (y; n \frown \underset{\smile}{\frown} (q \simeq \mathfrak{f})) \\ \mathfrak{0} \frown (n \frown \underset{\smile}{\frown} \mathfrak{f})\text{’} \end{array} \qquad (\beta$$

which we will prove by means of the propositions 202b

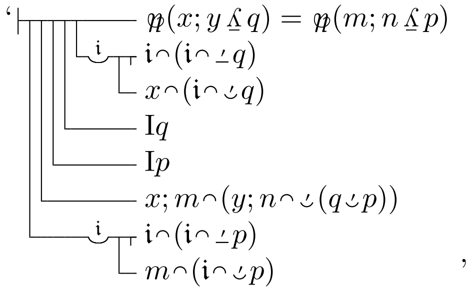


(γ



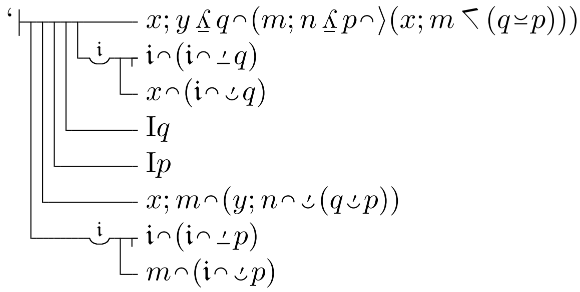
(δ

(γ) is traced back to the more general proposition



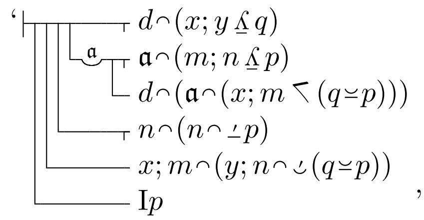
(ε

whose proof requires the proposition 202



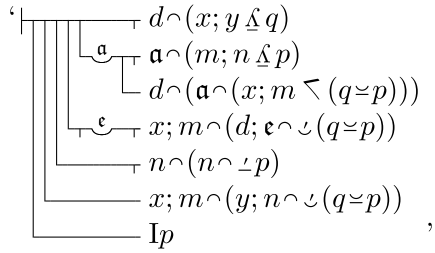
(ζ

According to (11), this resolves into the propositions (253) and | 202a



(η

We derive (η) from



(ϑ

by removing the subcomponent

$$\text{‘}\smile^{\mathfrak{e}}\!\!\top\; x;m\frown(d;\mathfrak{e}\frown\smile(q\asymp p))\text{’}$$

We prove (ϑ) by means of the proposition 203a

$$\text{‘}\vdash\begin{array}{l} c\frown(m;n\,\underline{\curlywedge}\,p) \\ \neg\; x;m\frown(d;c\frown\smile(q\asymp p)) \\ \neg\; n\frown(n\frown\perp p) \\ d\frown(x;y\,\underline{\curlywedge}\,q) \\ x;m\frown(y;n\frown\smile(q\asymp p)) \\ \mathrm{I}p \end{array}\text{’}\qquad (\iota$$

which leads back to the propositions (234) in the form

$$\text{‘}\vdash\begin{array}{l} m\frown(c\frown\smile p) \\ x;m\frown(d;c\frown\smile(q\asymp p)) \end{array}\text{’}$$

and

$$\text{‘}\vdash\begin{array}{l} c\frown(n\frown\smile p) \\ d\frown(x;y\,\underline{\curlywedge}\,q) \\ x;m\frown(y;n\frown\smile(q\asymp p)) \\ \mathrm{I}p \\ x;m\frown(d;c\frown\smile(q\asymp p)) \end{array}\text{’}\qquad (\kappa$$

203b

According to (243) we have

$$\text{‘}\vdash\begin{array}{l} d;c\frown(y;n\frown\smile(q\asymp p)) \\ \neg\; y;n\frown(d;c\frown\perp(q\asymp p)) \\ x;m\frown(y;n\frown\smile(q\asymp p)) \\ \mathrm{I}(q\asymp p) \\ x;m\frown(d;c\frown\smile(q\asymp p)) \end{array}\text{’}$$

By means of (244) we prove

$$\text{‘}\vdash\begin{array}{l} y\frown(y\frown\perp q) \\ y;n\frown(d;c\frown\perp(q\asymp p)) \\ d\frown(y\frown\smile q) \end{array}\text{’}\qquad (\lambda$$

with which we can remove the subcomponent

$$\text{‘}\top\; y;n\frown(d;c\frown\perp(q\asymp p))\text{’}$$

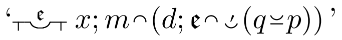
To begin with, we will draw the immediate consequences from our definition (P).

§159. *Construction*

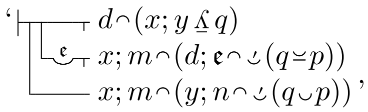
[proof to be added]

§160. *Analysis* 207a

Now the subcomponent

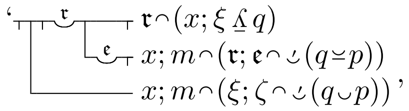


has to be eliminated (compare §158). This could be accomplished by means of the proposition

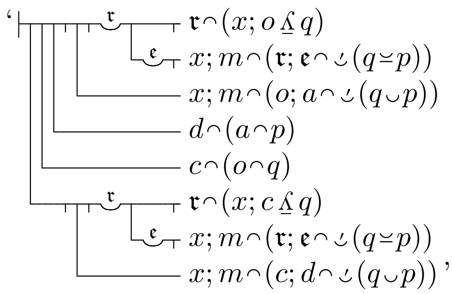


(α

which we prove by means of (257), replacing the function-marker ‘$F(\xi, \zeta)$’ by

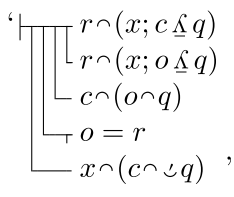


For this we require the proposition



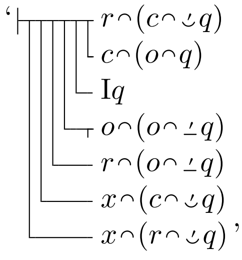
(β

This proposition is to be traced back to the proposition



(γ

which follows from | 207b



(δ

In order to prove (δ) we use proposition (243) in the form

$$‘\vdash \begin{array}{l} r \frown (c \frown \dot\smile q) \\ \top\, c \frown (r \frown \underline\smile q) \\ x \frown (c \frown \dot\smile q) \\ \mathrm{I}q \\ x \frown (r \frown \dot\smile q)\text{’} \end{array}$$

and show that, given our conditions, r cannot follow c in the q-series, since in that case, according to (242), r would belong to the q-series starting with o and therefore o would follow itself in the q-series.

§161. *Construction*

[proof to be added]

§162. *Analysis* 213b

In order to replace in (288) the subcomponent

$$‘\smile^{\mathfrak{i}}\!\top\, \mathfrak{i} \frown (\mathfrak{i} \frown \underline\smile q)\text{’}$$

by ‘$\top\, y \frown (y \frown \underline\smile q)$’, we exchange in (288) ‘q’ with ‘$y \frown \dot\smile q \bumpeq q$’. For the proposition

$$‘\vdash \begin{array}{l} \top\, i \frown (i \frown \underline\smile (y \frown \dot\smile q \bumpeq q)) \\ \top\, y \frown (y \frown \underline\smile q) \\ \mathrm{I}q \end{array}\text{’} \qquad (\alpha$$

is easily proven. In order to bring ‘q’ back into the proposition for ‘$y \frown \dot\smile q \bumpeq q$’, we apply the propositions (189)

$$‘\vdash \begin{array}{l} \not{\mathfrak{p}}(x; y \,\underline{\measuredangle}\, q) = \not{\mathfrak{p}}(x; y \,\underline{\measuredangle}\, (y \frown \dot\smile q \bumpeq q)) \\ \top\, y \frown (y \frown \underline\smile q) \\ \mathrm{I}q \end{array}\text{’} \qquad (\beta$$

and

$$‘\vdash \begin{array}{l} x; \not{\mathfrak{l}} \frown (y; n \frown \dot\smile (y \frown \dot\smile q \bumpeq q \veebar \mathfrak{f})) \\ x; \not{\mathfrak{l}} \frown (y; n \frown \dot\smile (q \veebar \mathfrak{f})) \end{array}\text{’} \qquad (\gamma$$

By means of (257), we prove the more general proposition resulting from (γ) by the replacement of ‘$\not{\mathfrak{l}}$’ by ‘m’ and ‘$\mathfrak{f}$’ by ‘p’. For this we need the proposition

$$‘\vdash \begin{array}{l} x; m \frown (a; o \frown \dot\smile (y \frown \dot\smile q \bumpeq q \veebar p)) \\ a \frown (y \frown \dot\smile q) \\ c \frown (o \frown p) \\ d \frown (a \frown q) \\ x; m \frown (d; c \frown \dot\smile (y \frown \dot\smile q \bumpeq q \veebar p)) \\ d \frown (y \frown \dot\smile q) \end{array}\text{’} \qquad (\delta$$

which follows from (209) and (197).

214a

§163. *Construction*

[proof to be added]

§164. *Analysis* 215a

In order to prove proposition (β) of §162 we need the propositions

$$`\vdash\!\!\begin{array}{l} x \frown (d \frown \smile(y \frown \smile q \asymp q)) \\ d \frown (y \frown \smile q) \\ x \frown (d \frown \smile q) \end{array}\text{'}, \qquad (\alpha$$

(194) and (189). We prove proposition (α) using (144).

§165. *Construction*

[proof to be added]

217

b) Proof of the proposition

$$`\vdash\!\!\begin{array}{l} n = \not{\mathfrak{p}}(\not{1}; n \,\underline{\measuredangle}\, \mathfrak{f}) \\ \not{0} \frown (n \frown \smile \mathfrak{f}) \end{array}\text{'}$$

and end of section K

§166. *Analysis* 217a

We prove proposition (δ) of §158 by means of (160). For this we require the proposition

$$`\vdash\!\!\begin{array}{l} \not{\mathfrak{p}}(\not{1}; n \,\underline{\measuredangle}\, \mathfrak{f}) = \not{\mathfrak{p}}\dot{\varepsilon}\left(\begin{array}{l} \varepsilon = \not{0} \\ \varepsilon \frown (n \frown \smile \mathfrak{f}) \end{array}\right) \\ \not{0} \frown (n \frown \smile \mathfrak{f}) \end{array}\text{'}, \qquad (\alpha$$

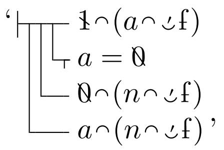
which follows using (IV a) and (96) from the propositions

$$`\vdash\!\!\begin{array}{l} a \frown (\not{1}; n \,\underline{\measuredangle}\, \mathfrak{f}) \\ a = \not{0} \\ \not{0} \frown (n \frown \smile \mathfrak{f}) \\ a \frown (n \frown \smile \mathfrak{f}) \end{array}\text{'}, \qquad (\beta$$

217b

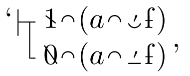
$$`\vdash\!\!\begin{array}{l} a = \not{0} \\ a \frown (n \frown \smile \mathfrak{f}) \\ a \frown (\not{1}; n \,\underline{\measuredangle}\, \mathfrak{f}) \end{array}\text{'} \qquad (\gamma$$

(β) is to be traced back to the proposition



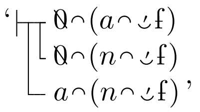
(δ

which can be derived from (242). We first obtain



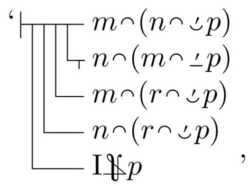
(ε

| It now turns on proving the proposition 218a



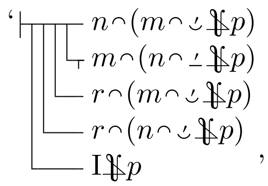
(ζ

This follows from the proposition

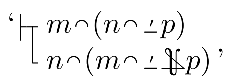


(η

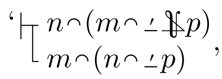
which is similar to (243) and can be derived from it. Writing (243) thus



it then remains to prove the propositions



(ϑ



(ι

and others similar to them, which can be accomplished by means of (123).

§167. *Construction*

[proof to be added]

221b

§168. *Analysis*

The last two transitions provide the basis for the removal of '*n*'. We now prove the proposition

$$‘\vdash\!\!\begin{array}{l} \overset{\mathfrak{e}}{\smile}\!\top\; x; \mathfrak{l}\frown(y; \mathfrak{e}\frown\smile(q\asymp\mathfrak{f})) \\ x\frown(y\frown\smile q) \end{array}’ \qquad (\alpha$$

by means of (144), requiring for this the proposition

$$‘\vdash\!\!\begin{array}{l} \overset{\mathfrak{e}}{\smile}\!\top\; x; \mathfrak{l}\frown(a; \mathfrak{e}\frown\smile(q\asymp\mathfrak{f})) \\ d\frown(a\frown q) \\ \overset{\mathfrak{e}}{\smile}\!\top\; x; \mathfrak{l}\frown(d; \mathfrak{e}\frown\smile(q\asymp\mathfrak{f})) \end{array}’ \qquad (\beta$$

which we derive from (209).

§169. *Construction*

[proof to be added]

⋮

$$\vdash\!\!\begin{array}{l} c\frown(o\frown\mathfrak{f}) \\ \overset{\mathfrak{e}}{\smile}\!\top\; x; \mathfrak{l}\frown(a; \mathfrak{e}\frown\smile(q\asymp\mathfrak{f})) \\ d\frown(a\frown q) \\ x; \mathfrak{l}\frown(d; c\frown\smile(q\asymp\mathfrak{f})) \end{array} \qquad (\beta$$

⋮

222b

§170. *Analysis*

The subcomponent in (321) will now be eliminated. To this end, we prove the proposition

$$‘\vdash\!\!\begin{array}{l} \wp(x; y \mathbin{\underline{\measuredangle}} q) = \emptyset \\ x\frown(y\frown\smile q) \\ y\frown(y\frown\underline{\smile} q) \\ \mathrm{I}q \end{array}’ \qquad (\alpha$$

using (97), (271), (265) and the proposition

$$‘\vdash\!\!\begin{array}{l} x\frown(y\frown\smile q) \\ d\frown(x; y \mathbin{\underline{\measuredangle}} q) \end{array}’ \qquad (\beta$$

which follows from (269) and (270) using the proposition

$$‘\vdash\!\!\begin{array}{l} x\frown(y\frown\smile q) \\ x\frown(d\frown\smile q) \\ d\frown(y\frown\smile q) \end{array}’ \qquad (\gamma$$

which is proven using (144).

§171. *Construction*

[proof to be added]

224

Λ. Proof of the proposition

$$`\begin{array}{l}\vdash\!\!\top\!\!\overset{\mathfrak{A}}{\smile}\!\!\overset{\mathfrak{q}}{\smile}\!\!\top\ \grave{\varepsilon}(-\,\varepsilon\frown u) = \mathfrak{A} \mathbin{\underline{\measuredangle}} \mathfrak{q} \\ \quad\llcorner\!\!-\!\!-\!\!-\!\!-\ 0\frown(\wp u\frown\smile \mathfrak{f})\end{array}\text{ ,}$$

224a

§172. *Analysis*

We attempt to express in words the proposition to be proven like this:

> "If the cardinal number of a concept is finite, then the objects that fall under it can be ordered into a simple series running from a specific object to a specific object."

This expression is not perfect insofar as according to it the proposition seems not to apply to the cardinal number Zero. However, we may take a series-forming relation such that no object belongs to its series running from Δ to Θ by the condition never being met that is demanded by definition (P) on there being an object belonging to the series running from Δ to Θ.

By (314) we have

$$`\begin{array}{l}\vdash\!\top\ \wp u = \wp(1;\, \wp u \mathbin{\underline{\measuredangle}} \mathfrak{f}) \\ \ \llcorner\ 0\frown(\wp u\frown\smile \mathfrak{f})\end{array}\text{ ,}$$

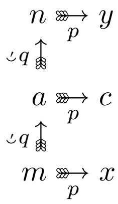
According to this, there is a relation which maps the $(1; \wp u \mathbin{\underline{\measuredangle}} \mathfrak{f})$-concept into the u-concept | while its converse maps the latter into the former. This will be our p-relation. We now show that we can take the $(\mathfrak{U}p \smile \mathfrak{f} \smile p)$-relation as series-forming. It is first to be shown that every object falling under the u-concept belongs to the $(\mathfrak{U}p \smile \mathfrak{f} \smile p)$-series going from x to y, where 1 stands to x, and $\wp u$ stands to y in the p-relation. More generally we write 'm' instead of '1', 'n' instead of '$\wp u$' and 'q' instead of '$\mathfrak{f}$' and prove the proposition 224b

$$`\begin{array}{l}\vdash\!\top\!\top\!\top\!\top\!\top\ c\frown(x;\, y \mathbin{\underline{\measuredangle}} (\mathfrak{U}p\smile q\smile p)) \\ \quad\llcorner\ c\frown u \\ \quad\llcorner\ m\frown(x\frown p) \\ \quad\llcorner\ m;\, n \mathbin{\underline{\measuredangle}} q\frown(u\frown\rangle p) \\ \quad\llcorner\ u\frown(m;\, n \mathbin{\underline{\measuredangle}} q\frown\rangle \mathfrak{U}p) \\ \quad\llcorner\ n\frown(y\frown p)\end{array}\text{ ,} \qquad (\alpha$$

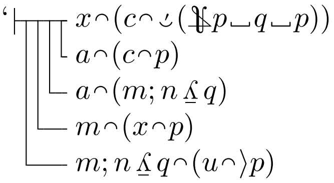
which, using (8), follows from

$$`\begin{array}{l}\vdash\!\top\!\top\!\top\!\top\!\top\!\top\ c\frown(x;\, y \mathbin{\underline{\measuredangle}} (\mathfrak{U}p\smile q\smile p)) \\ \quad\llcorner\ c\frown(a\frown \mathfrak{U}p) \\ \quad\llcorner\ a\frown(m;\, n \mathbin{\underline{\measuredangle}} q) \\ \quad\llcorner\ m\frown(x\frown p) \\ \quad\llcorner\ m;\, n \mathbin{\underline{\measuredangle}} q\frown(u\frown\rangle p) \\ \quad\llcorner\ \mathrm{I}\mathfrak{U}p \\ \quad\llcorner\ n\frown(y\frown p)\end{array}\text{ ,} \qquad (\beta$$

| The following diagram will assist the understanding 225a

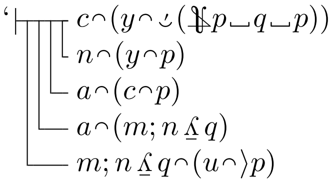


The propositions needed in order to prove (β) include amongst others the following:



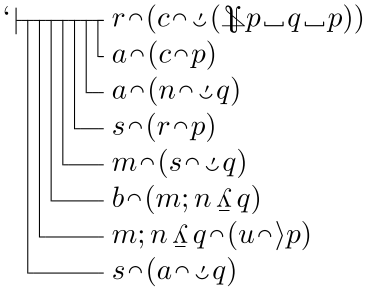
, (γ

and



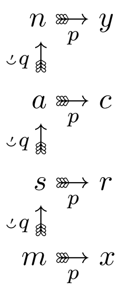
, (δ

which we derive from



, (ε

by first letting r coincide with x, s with m and b with a, and then c with y, a with n and b with s, and finally writing 'a' for 's' and 'c' for 'r'. For this, compare the following diagram



| To derive (ε), we employ (152), replacing the function-marker ‘$F(\xi)$’ by 225b

$$
`\,\smile^{\mathfrak{e}}\!\!\begin{array}{l} r\frown(\mathfrak{e}\frown\smile(\not\!\!\backslash p \llcorner q \llcorner p)) \\ \xi\frown(\mathfrak{e}\frown p) \\ \xi\frown(n\frown\smile q) \end{array}\quad ,
$$

In doing so, as in the proof of (186), the propositions (183) and (185) are to be used, thus introducing the subcomponent

$$
`\!\!- m; n \underline{\measuredangle}\, q\frown(u\frown\rangle p)\text{'}
$$

§173. *Construction*

[proof to be added]

227b

§174. *Analysis*

In order to eliminate the subcomponent

$$
`\top\, y\frown(y\frown\underline{\prime}(\not\!\!\backslash p\llcorner q\llcorner p))\text{'}
$$

we prove the proposition

$$
`\vdash\begin{array}{l} m\frown(n\frown\underline{\prime} q) \\ \mathrm{I}\not\!\!\backslash p \\ x\frown(y\frown\underline{\prime}(\not\!\!\backslash p\llcorner q\llcorner p)) \\ m\frown(x\frown p) \\ n\frown(y\frown p) \end{array}\quad , \qquad (\alpha
$$

which, using (177), we trace back to

$$
`\vdash\begin{array}{l} m\frown(n\frown t) \\ x\frown(y\frown(\not\!\!\backslash p\llcorner t\llcorner p)) \\ m\frown(x\frown p) \\ \mathrm{I}\not\!\!\backslash p \\ n\frown(y\frown p) \end{array}\quad , \qquad (\beta
$$

We show that there are objects s and a such that s stands to x and a to y in the p-relation and hence that a follows after s in the q-series. From the single-valuedness of the converse of the p-relation it follows that s coincides

with m | and a with n, thus that n follows after m in the q-series. The 228a
subcomponent

$$\text{‘}— \mathrm{I}(\between p \smile q \smile p)\text{’}$$

is also to be removed. This is easily done by means of (17).

§175. *Construction*

[proof to be added]

229b

§176. *Analysis*

We now have to prove the proposition (α) of §172. The proposition | 230a

$$\begin{array}{l}\text{‘}\vdash c\frown u \\ \quad c\frown(x;y \lneq (\between p\smile q \smile p)) \\ \quad m\frown(x\frown p) \\ \quad u\frown(m;n \lneq q\frown\rangle\between p) \\ \quad n\frown(y\frown p) \\ \quad \neg\, n\frown(n\frown\underset{\cdot}{\smile} q) \\ \quad m;n \lneq q\frown(u\frown\rangle p) \\ \quad \mathrm{I}q \text{’}\end{array} \qquad (\alpha$$

remains to be derived. The latter is to be derived using (179) from the propositions

$$\begin{array}{l}\text{‘}\vdash \neg\, x\frown(c\frown\underset{\cdot}{\smile}(\between p\smile q\smile p)) \\ \quad m\frown(x\frown p) \\ \quad \mathrm{I}\between p \\ \quad \forall\mathfrak{r}\; \neg\, m\frown(\mathfrak{r}\frown\underset{\cdot}{\smile} q) \\ \quad\quad \mathfrak{r}\frown(c\frown p) \text{’}\end{array} \qquad (\beta$$

and

$$\begin{array}{l}\text{‘}\vdash \neg\, c\frown(y\frown\underset{\cdot}{\smile}(\between p\smile q\smile p)) \\ \quad n\frown(y\frown p) \\ \quad \forall\mathfrak{r}\; \neg\, \mathfrak{r}\frown(n\frown\underset{\cdot}{\smile} q) \\ \quad\quad \mathfrak{r}\frown(c\frown p) \\ \quad \mathrm{I}\between p \text{’}\end{array} \qquad (\gamma$$

by inferring from the single-valuedness of the $\between p$-relation that the same object which stands to c in the p-relation belongs both to the q-series starting with m and the one ending with n. Instead of (β) and (γ) we first prove the propositions that, while having the same subcomponents, have as supercomponents

$$\text{‘}\neg\, x\frown(c\frown(\between p\smile\underset{\cdot}{\smile} q\smile p))\text{’}$$

and

$$‘\top\, c \frown (y \frown (\not{\mathfrak{F}}p \smile \dot{\smile} q \smile p))’$$

Using proposition (180) we can move to (β). In order to arrive at (γ) we need the similar proposition

$$‘\begin{array}{l} \vdash\!\!\top\!\!\top\!\!\top\; c \frown (y \frown (\not{\mathfrak{F}}p \smile \dot{\smile} q \smile p)) \\ \quad\quad\;\llcorner\; n \frown (y \frown p) \\ \quad\;\;\llcorner\; c \frown (y \frown \dot{\smile} (\not{\mathfrak{F}}p \smile q \smile p)) \\ \;\llcorner\; \mathrm{I}\not{\mathfrak{F}}p \end{array}’ \qquad (\delta$$

which we derive from (177).

| **§177.** *Construction* 230b

[proof to be added]

| **§178.** *Analysis* 237b

We remove the subcomponent

$$‘\top\!\overset{\mathfrak{a}}{\smile}\!\top\; \mathfrak{a} \frown u’$$

by supplying a series-forming relation such that no object belongs to its series running from an object to an object, as was stated in §172. Such a relation is equality since every object in the series of this relation follows itself.

§179. *Construction*

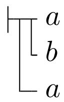
[proof to be added]

239

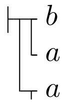
Appendices

1. Table of the basic laws

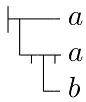
and propositions immediately following from them | 239a



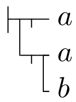
(I (§18)



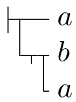
(I a (§49)



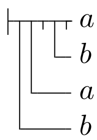
(I b (§49)



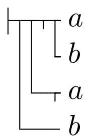
(I c (§49)



(I d (§49)

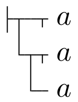


(I e (§49)

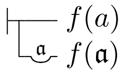


(I f (§49)

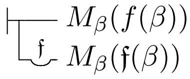
239b



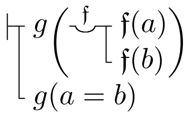
(I g (§49)



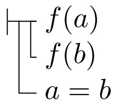
(II a (§20)



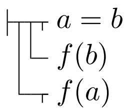
(II b (§25)



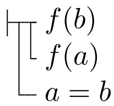
(III (§20)



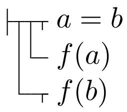
(III a (§50)



(III b (§50)



(III c (§50)



(III d (§50)

240a

$\vdash a = a$ (III e (§50)

——— • ———

$\vdash\!\!\top\; b = a$
$\quad\llcorner\; a = b$ (III f (§50)

——— • ———

$\vdash\!\!\top\, (\text{—}\, a) = (\top\, a)$ (III g (§50)

——— • ———

$\vdash\!\!\top\; f(a) = f(b)$
$\quad\llcorner\; a = b$ (III h (§50)

——— • ———

$\vdash (\text{—}\, a = b) = (a = b)$ (III i (§50)

——— • ———

$\vdash\!\!\top\; (\text{—}\, a) = (\text{—}\, b)$
$\quad\llcorner\; (\text{—}\, a) = (\top\, b)$ (IV (§18)

——— • ———

240b

$\vdash\!\!\top\!\!\top\; (\text{—}\, a) = (\text{—}\, b)$
$\quad\quad\llcorner\!\top\; b$
$\quad\quad\quad\llcorner\; a$
$\quad\llcorner\!\top\; a$
$\quad\quad\llcorner\; b$ (IV a (§51)

——— • ———

$\vdash (\text{—}\, a) = (\top\!\top\, a)$ (IV b (§51)

——— • ———

$\vdash\!\!\top\; f(\text{—}\, a)$
$\quad\llcorner\; f(\top\!\top\, a)$ (IV c (§51)

——— • ———

$\vdash\!\!\top\; f(\top\!\top\, a)$
$\quad\llcorner\; f(\text{—}\, a)$ (IV d (§51)

——— • ———

$\vdash (a = b) = (b = a)$ (IV e (§51)

——— • ———

$$\vdash (\grave{\varepsilon} f(\varepsilon) = \grave{\alpha} g(\alpha)) = (\overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = g(\mathfrak{a})) \qquad \text{(V (§20)}$$

——— • ———

$$\vdash F(\grave{\varepsilon} f(\varepsilon)) = F(\grave{\alpha} g(\alpha)) \;\; \text{if} \;\; \overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = g(\mathfrak{a}) \qquad \text{(V a (§52)}$$

——— • ———

$$\vdash f(a) = g(a) \;\; \text{if} \;\; \grave{\varepsilon} f(\varepsilon) = \grave{\alpha} g(\alpha) \qquad \text{(V b (§52)}$$

——— • ———

$$\vdash a = \backslash \grave{\varepsilon}(a = \varepsilon) \qquad \text{(VI (§18)}$$

——— • ———

$$\vdash a = \backslash \grave{\varepsilon} f(\varepsilon) \;\; \text{if} \;\; \overset{\mathfrak{a}}{\smile}\, f(\mathfrak{a}) = (a = \mathfrak{a}) \qquad \text{(VI a (§52)}$$

2. Table of definitions 240

$$\Vdash \backslash \grave{\alpha}\left(\overset{\mathfrak{g}}{\smile} \left[\mathfrak{g}(a) = \alpha \;\; \text{and} \;\; u = \grave{\varepsilon}\mathfrak{g}(\varepsilon) \right] \right) = a \frown u \qquad \text{(A}$$

(Relation of an object falling within the extension of a concept. §34, p. 53.)[1] 240b

$$\Vdash \grave{\alpha}\grave{\varepsilon}\left(\overset{\mathfrak{r}}{\smile} \left[\varepsilon \frown (\mathfrak{r} \frown p) \;\; \text{and} \;\; \mathfrak{r} \frown (\alpha \frown q) \right] \right) = p \llcorner\!\lrcorner q \qquad \text{(B}$$

(Composite relation. §54, p. 72.)

$$\Vdash \left(\overset{\mathfrak{e}}{\smile}\overset{\mathfrak{d}}{\smile}\overset{\mathfrak{a}}{\smile} \left[\mathfrak{d} = \mathfrak{a} \;\; \text{if} \;\; \mathfrak{e} \frown (\mathfrak{a} \frown p) \;\; \text{and} \;\; \mathfrak{e} \frown (\mathfrak{d} \frown p) \right] \right) = \mathrm{I}p \qquad (\Gamma$$

(Single-valuedness of a relation. §37, p. 55.) 241a

$$\Vdash \grave{\alpha}\grave{\varepsilon}\left[\overset{\mathfrak{d}}{\smile} \left[\mathfrak{d} \frown \varepsilon \;\; \text{and} \;\; \overset{\mathfrak{a}}{\smile} \left(\mathfrak{a} \frown \alpha \;\; \text{if} \;\; \mathfrak{d} \frown (\mathfrak{a} \frown p) \right) \right] \;\; \text{and} \;\; \mathrm{I}p \right] = \rangle p \qquad (\Delta$$

(Mapping-into by a relation. §38, p. 56.[a])

$$\Vdash \grave{\alpha}\grave{\varepsilon}(\alpha \frown (\varepsilon \frown p)) = \mathfrak{X}p \qquad \text{(E}$$

(Converse of a relation. §39, p. 57.[b])

[1] These short hints in words which I add to the concept-script definitions are not exhaustive and make no claim to be of the strictest precision.

[a] *Translators' Note*: Typo in Frege, noted by Thiel: '57' instead of '56'.

[b] *Translators' Note*: Typo in Frege, noted by Thiel: '56' instead of '57'.

$$\Vdash \dot{\varepsilon}\left(\begin{array}{l} \overset{\mathfrak{q}}{\smile} \top\!\top\ \varepsilon \frown (u \frown \rangle \mathfrak{q}) \\ \qquad \llcorner\ u \frown (\varepsilon \frown \rangle \natural \mathfrak{q}) \end{array} \right) = \wp u \qquad \text{(Z}$$

(The cardinal number of a concept; i.e., the cardinal number of objects falling under a concept. §40, p. 57.)

$$\Vdash \dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l} \overset{\mathfrak{u}}{\smile}\overset{\mathfrak{a}}{\smile}\top\!\top\!\top\ \wp\mathfrak{u} = \alpha \\ \qquad \llcorner\ \mathfrak{a} \frown \mathfrak{u} \\ \qquad \llcorner\ \wp\dot{\varepsilon}\left(\begin{array}{l} \top\ \varepsilon = \mathfrak{a} \\ \llcorner\ \varepsilon \frown \mathfrak{u} \end{array} \right) = \varepsilon \end{array} \right] = \mathfrak{f} \qquad \text{(H}$$

(Relation of a cardinal number to the one immediately following. §43, p. 58.)

$$\Vdash \wp\dot{\varepsilon}(\top\ \varepsilon = \varepsilon) = \emptyset \qquad \text{(}\Theta$$

(The cardinal number Zero. §41, p. 58.)

$$\Vdash \wp\dot{\varepsilon}(\varepsilon = \emptyset) = 1 \qquad \text{(I}$$

(The cardinal number One. §42, p. 58.) 241b

$$\Vdash \dot{\alpha}\dot{\varepsilon}\left[\begin{array}{l} \overset{\mathfrak{F}}{\smile}\top\!\top\!\top\ \mathfrak{F}(\alpha) \\ \qquad \llcorner \overset{\mathfrak{a}}{\smile}\top\ \mathfrak{F}(\mathfrak{a}) \\ \qquad\qquad \llcorner\ \varepsilon \frown (\mathfrak{a} \frown q) \\ \qquad \llcorner \overset{\mathfrak{d}}{\smile}\overset{\mathfrak{a}}{\smile}\top\ \mathfrak{F}(\mathfrak{a}) \\ \qquad\qquad \llcorner\ \mathfrak{d} \frown (\mathfrak{a} \frown q) \\ \qquad \llcorner\ \mathfrak{F}(\mathfrak{d}) \end{array} \right] = \,\underline{\smile}\, q \qquad \text{(K}$$

(The following of an object after an object in the series of a relation. §45, p. 60.)

$$\Vdash \dot{\alpha}\dot{\varepsilon}\left(\begin{array}{l} \top\ \alpha = \varepsilon \\ \llcorner\!\top\ \varepsilon \frown (\alpha \frown \underline{\smile} q) \end{array} \right) = \smile q \qquad \text{(}\Lambda$$

(The relation of an object belonging to the series of a relation starting with an object. §46, p. 60.)

$$\Vdash \wp(\emptyset \frown \natural \smile \mathfrak{f}) = \infty \qquad \text{(M}$$

(The cardinal number Endlos. §122, p. 150.)

$$\Vdash \dot{\alpha}\dot{\varepsilon}\left(\begin{array}{l} \top\!\top\ \varepsilon \frown (\alpha \frown q) \\ \quad \llcorner\ \alpha \frown u \end{array} \right) = u \frown q \qquad \text{(N}$$

(§138, p. 171.)

$$\Vdash \dot{\varepsilon}(o \frown (a \frown \varepsilon)) = o;a \qquad \text{(}\Xi$$

(The pair. §144, p. 179.) 241

$$\Vdash \grave{\alpha}\acute{\varepsilon}\left[\begin{array}{l} \top\!\!\smile^{\mathfrak{a}}\!\!\smile^{\mathfrak{o}}\!\!\top\!\!\smile^{\mathfrak{d}}\!\!\top\!\!\smile^{\mathfrak{c}}\!\!\top\!\!\top\; \mathfrak{c}\frown(\mathfrak{o}\frown p) \\ \qquad\qquad\qquad\quad\; \varepsilon = \mathfrak{c};\mathfrak{d} \\ \qquad\qquad\qquad\quad\; \mathfrak{d}\frown(\mathfrak{a}\frown q) \\ \qquad\qquad\qquad\quad\; \alpha = \mathfrak{o};\mathfrak{a} \end{array}\right] = p\,\underline{\smile}\,q \qquad (\mathrm{O}$$

(Coupling of a relation with a relation. §144, p. 179.)

$$\Vdash \grave{\alpha}\acute{\varepsilon}(A\frown(\varepsilon;\alpha\frown\smile t)) = A \lessdot t \qquad (\Pi$$

(§144, p. 179.)

$$\Vdash \acute{\varepsilon}\left[\begin{array}{l} \top\!\!\smile^{\mathfrak{n}}\!\!\smile^{\mathfrak{m}}\!\!\top\!\!\top\!\!\top\; \varepsilon\frown(\mathfrak{n}\frown\smile q) \\ \qquad\qquad\quad\; \mathfrak{m}\frown(\varepsilon\frown\smile q) \\ \qquad\qquad\; \top\; \mathfrak{n}\frown(\mathfrak{n}\frown\smile q) \\ \qquad\qquad\quad\; A = \mathfrak{m};\mathfrak{n} \\ \qquad\qquad\quad\; \mathrm{I}q \end{array}\right] = A \,\underline{\lessdot}\, q \qquad (\mathrm{P}$$

(The circumstance that an object belongs to a series running from an object to an object. §158, p. 201.)

3. Table of the important theorems 242

$$\vdash f(a) = a \frown \dot{\varepsilon} f(\varepsilon)$$ (1

—— • ——

$$\begin{array}{l} \vdash\!\!\!\top F(a \frown \dot{\varepsilon} f(\varepsilon)) \\ \quad\llcorner F(f(a)) \end{array}$$ (77

—— • ——

$$\begin{array}{l} \vdash\!\!\!\top F(f(a)) \\ \quad\llcorner F(a \frown \dot{\varepsilon} f(\varepsilon)) \end{array}$$ (82

—— • ——

$$\vdash f(a,b) = a \frown (b \frown \dot{\alpha}\dot{\varepsilon} f(\varepsilon, \alpha))$$ (2

—— • ——

$$\begin{array}{l} \vdash\!\!\!\top\!\!\top F(f(a,b)) \\ \quad\llcorner F(a \frown (b \frown q)) \\ \quad\llcorner \dot{\alpha}\dot{\varepsilon} f(\varepsilon, \alpha) = q \end{array}$$ (6

—— • ——

$$\begin{array}{l} \vdash\!\!\!\top\!\!\top F(a \frown (b \frown q)) \\ \quad\llcorner F(f(a,b)) \\ \quad\llcorner \dot{\alpha}\dot{\varepsilon} f(\varepsilon, \alpha) = q \end{array}$$ (10

—— • ——

$$\begin{array}{l} \vdash\!\!\!\top f(a,b) \\ \quad\llcorner F(a \frown (b \frown \dot{\alpha}\dot{\varepsilon} f(\varepsilon, \alpha)) \end{array}$$ (33

—— • ——

$$\begin{array}{l} \vdash\!\!\!\top F(a \frown (b \frown \dot{\alpha}\dot{\varepsilon} f(\varepsilon, \alpha))) \\ \quad\llcorner F(f(a,b)) \end{array}$$ (36

—— • ——

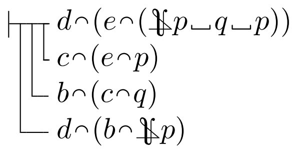
$$\begin{array}{l} \vdash\!\!\!\top F(\top\, a \frown \dot{\varepsilon}(\top\, f(\varepsilon))) \\ \quad\llcorner F(\text{—}\, f(a)) \end{array}$$ (58

—— • ——

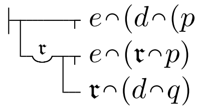
$$\begin{array}{l} \vdash\!\!\!\top\!\!\top d \frown (m \frown (p \sqcup q)) \\ \quad\llcorner e \frown (m \frown q) \\ \quad\llcorner d \frown (e \frown p) \end{array}$$ (5

If an object (d) stands in the (p-)relation to a second (e) and if the second object (e) stands in the (q-)relation to a third (m), then the first object stands to the third in the | relation composed of the first and the second.[1] 242b

[1]The translations that I append to the concept-script propositions reflect their principal content, but do not always exhaust the whole content.



(174



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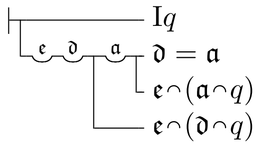

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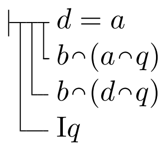
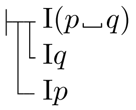


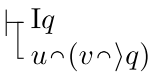
(13

If a relation is single-valued and an object (b) stands to a second (d) and a third (a) in this relation, then the second (d) and the third (a) will coincide.



(17

A relation composed of two relations is single-valued if these two relations are. | 243a



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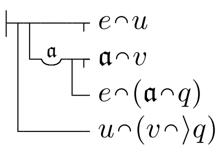

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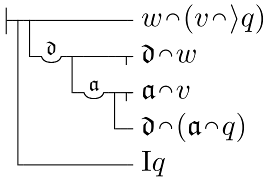

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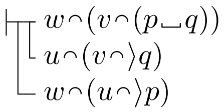


(19

If a relation maps one concept into a second, and this second concept is mapped into a third by a second (q-)relation, then the relation composed of the first and second relation maps the first concept into the third.

$$\vdash\!\!\begin{array}{l} F(a \frown (r \frown \mathfrak{U} q)) \\ F(r \frown (a \frown q)) \end{array} \qquad (22$$

——— • ———

$$\vdash\!\!\begin{array}{l} F(r \frown (a \frown q)) \\ F(a \frown (r \frown \mathfrak{U} q)) \end{array} \qquad (23$$

——— • ———

$$\vdash \mathfrak{U}(p \smile q) = \mathfrak{U} q \smile \mathfrak{U} p \qquad (24$$

The converse of a relation which is composed of a first and second, is composed of the converse of the second and the converse of the first.

$$\vdash\!\!\begin{array}{l} \not{p}w = \not{p}z \\ \overset{\mathfrak{q}}{\smile}\; w \frown (z \frown \rangle \mathfrak{q}) \\ \quad z \frown (w \frown \rangle \mathfrak{U} \mathfrak{q}) \end{array} \qquad (49$$

The cardinal number of objects falling under a first (w-)concept does not coincide with the cardinal number of objects falling under a second (z-)concept if there is no relation that maps the first into the second and whose converse also maps the second into the first.

$$\vdash\!\!\begin{array}{l} \not{p}u = \not{p}v \\ u \frown (v \frown \rangle q) \\ v \frown (u \frown \rangle \mathfrak{U} q) \end{array} \qquad (32$$

The cardinal number of the objects falling under the (u-)concept coincides with the cardinal number of those falling under a second (v-)concept if a relation maps the first into the second concept whose converse maps the second into the first.

$$\vdash\!\!\begin{array}{l} \not{p}u = \not{p}v \\ \overset{\mathfrak{a}}{\smile}\; (\text{—}\, \mathfrak{a} \frown u) = (\text{—}\, \mathfrak{a} \frown v) \end{array} \qquad (96$$

——— • ———

$$\vdash\!\!\begin{array}{l} e \frown (a \frown \mathfrak{f}) \\ \overset{\mathfrak{u}}{\smile}\overset{\mathfrak{a}}{\smile}\; \not{p}\mathfrak{u} = a \\ \quad \mathfrak{a} \frown \mathfrak{u} \\ \quad \not{p}\dot{\varepsilon}\left(\begin{array}{l} \varepsilon = \mathfrak{a} \\ \varepsilon \frown \mathfrak{u} \end{array}\right) = e \end{array} \qquad (68$$

——— • ———

$$\vdash \mathrm{I}\mathfrak{f} \qquad (71$$

The relation of a cardinal number to that immediately following it in the cardinal number series is single-valued.

$$\vdash \mathrm{I}\mathfrak{U}\mathfrak{f} \qquad (89$$

The relation of a cardinal number to that immediately preceding it in the cardinal number series is single-valued.

$$\begin{array}{l}\vdash\; m \frown (n \frown \mathfrak{f}) \\ \quad c \frown u \\ \quad \wp\dot{\varepsilon}\left(\begin{array}{l} \varepsilon = c \\ \varepsilon \frown u \end{array}\right) = m \\ \quad \wp u = n \end{array} \qquad (101$$

——— • ———

244a

$$\begin{array}{l}\vdash\; a \frown u \\ \quad \wp u = \mathbb{0} \end{array} \qquad (94$$

If Zero is the cardinal number of objects that fall under a concept, then no object falls under the concept.

$$\vdash\; c \frown (\mathbb{0} \frown \mathfrak{f}) \qquad (108$$

In the cardinal number series nothing immediately precedes Zero.

$$\begin{array}{l}\vdash\; \wp u = \mathbb{0} \\ \quad \smile^{\mathfrak{a}}\; \mathfrak{a} \frown u \end{array} \qquad (97$$

If no object falls under a concept, then Zero is the cardinal number of the objects falling under that concept.

$$\begin{array}{l}\vdash\; \smile^{\mathfrak{a}}\; \mathfrak{a} \frown (a \frown \mathfrak{f}) \\ \quad a = \mathbb{0} \\ \quad \smile^{\mathfrak{u}}\; \wp \mathfrak{u} = a \end{array} \qquad (107$$

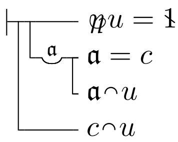
For every cardinal number distinct from Zero, there is one immediately preceding it in the cardinal number series.

$$\begin{array}{l}\vdash\; \wp v = \mathbb{0} \\ \quad \wp u = \mathbb{0} \\ \quad \smile^{\mathfrak{a}}\; \mathfrak{a} \frown u \\ \quad \mathfrak{a} \frown v \end{array} \qquad (99$$

If Zero is the cardinal number of the objects falling under a first concept, then Zero is also the cardinal number of the objects that fall under a concept subordinated to the first.

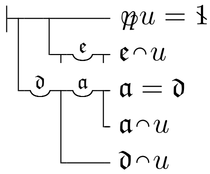
$$\begin{array}{l}\vdash\; d = a \\ \quad a \frown u \\ \quad \wp u = \mathbb{1} \\ \quad d \frown u \end{array} \qquad (117$$

If One is the cardinal number of objects falling under a concept and if
a first object falls under this concept and likewise | a second, then these 244b
objects coincide.



(121

If an object falls under a concept and if every object that falls under this concept coincides with the former, then One is the cardinal number of the objects falling under the concept.



(122

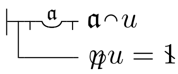
One is the cardinal number of the objects falling under a concept if there is an object that falls under it and every object that falls under the concept coincides with any object that falls under it.

$$\vdash \mathfrak{0} \frown (1 \frown \mathfrak{f}) \qquad (110$$

The cardinal number One immediately follows the cardinal number Zero in the cardinal number series.

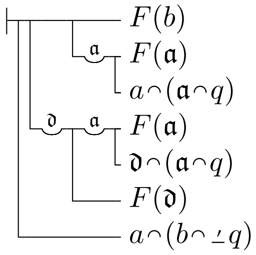
$$\vdash\!\!\top\; \mathfrak{0} = 1 \qquad (111$$

The cardinal number Zero is distinct from the cardinal number One.



(113

If One is the cardinal number of objects falling under a concept, then there is one object falling under the concept. | 245a



(123

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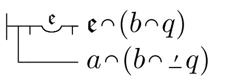


(124

If an object follows an object in a series, then there is an object which stands to the former in the series-forming relation.

$$\vdash\!\!\top\; a \frown (\mathfrak{0} \frown \underline{\cdot}\, \mathfrak{f}) \qquad (126$$

Nothing precedes the cardinal number Zero in the cardinal number series.

$$
\begin{array}{l}
\vdash a \frown (b \frown \underline{'} q) \\
\quad \mathfrak{F}:\ \mathfrak{F}(b) \\
\quad\quad \mathfrak{a}:\ \mathfrak{F}(\mathfrak{a}) \\
\quad\quad\quad a \frown (\mathfrak{a} \frown q) \\
\quad\quad \mathfrak{d}\ \mathfrak{a}:\ \mathfrak{F}(\mathfrak{a}) \\
\quad\quad\quad \mathfrak{d} \frown (\mathfrak{a} \frown q) \\
\quad\quad\quad \mathfrak{F}(\mathfrak{d})
\end{array}
\qquad (127
$$

——— • ———

$$
\begin{array}{l}
\vdash d \frown (a \frown \underline{'} q) \\
\quad d \frown (a \frown q)
\end{array}
\qquad (131
$$

A first object precedes a second in a series if it stands to it in the series-forming relation.

$$
\begin{array}{l}
\vdash a \frown (m \frown \underline{'} q) \\
\quad e \frown (m \frown q) \\
\quad a \frown (e \frown \underline{'} q)
\end{array}
\qquad (133
$$

If an object follows a second in a series and stands to a third in the series-forming relation, then the third follows the second in this series. | 245b

$$
\begin{array}{l}
\vdash x \frown (y \frown \underline{'} q) \\
\quad x \frown (d \frown \underline{'} q) \\
\quad d \frown (y \frown \underline{'} q)
\end{array}
\qquad (275
$$

If an object follows a second in a series and precedes a third in it, then the third also follows the second in this series.

$$
\begin{array}{l}
\vdash x \frown (y \frown (\mathfrak{Y} p \sqcup \underline{'} q \sqcup p)) \\
\quad \mathrm{I} \mathfrak{Y} p \\
\quad x \frown (y \frown \underline{'} (\mathfrak{Y} p \sqcup q \sqcup p))
\end{array}
\qquad (177
$$

——— • ———

$$
\begin{array}{l}
\vdash d \frown (c \frown \underline{'} q) \\
\quad d \frown (a \frown q) \\
\quad a \frown (c \frown \underline{'} q)
\end{array}
\qquad (129
$$

——— • ———

$$
\begin{array}{l}
\vdash n \frown (m \frown \underline{'} \mathfrak{Y} p) \\
\quad m \frown (n \frown \underline{'} p)
\end{array}
\qquad (302
$$

——— • ———

$$
\begin{array}{l}
\vdash m \frown (n \frown \underline{'} p) \\
\quad n \frown (m \frown \underline{'} \mathfrak{Y} p)
\end{array}
\qquad (299
$$

An object follows a second in the series of a relation if the second follows the first in the series of the converse of the relation.

$$\vdash \begin{array}{l} F\left(\begin{array}{l} c = a \\ a \frown (c \frown \dot{\bot} q)\end{array}\right) \\ F(a \frown (c \frown \dot{\smile} q)) \end{array} \qquad (130$$

—— • ——

$$\vdash \begin{array}{l} x \frown (y \frown \dot{\bot} q) \\ y = x \\ x \frown (y \frown \dot{\smile} q) \end{array} \qquad (200$$

If an object belongs to a series starting with a second, then it either coincides with it or follows it in this series.

$$\vdash \begin{array}{l} x \frown (y \frown \dot{\bot} q) \\ x \frown (d \frown \dot{\bot} q) \\ d \frown (y \frown \dot{\smile} q) \end{array} \qquad (276$$

—— • ——

246a

[add formula]

$$\vdash \begin{array}{l} x \frown (y \frown \dot{\bot} q) \\ x \frown (d \frown \dot{\smile} q) \\ d \frown (y \frown \dot{\bot} q) \end{array} \qquad (280$$

—— • ——

$$\vdash \begin{array}{l} F(b) \\ F(a) \\ \overset{\mathfrak{d}}{\smile}\ \overset{\mathfrak{a}}{\smile}\ F(\mathfrak{a}) \\ \mathfrak{d} \frown (\mathfrak{a} \frown q) \\ F(\mathfrak{d}) \\ a \frown (b \frown \dot{\smile} q) \end{array} \qquad (144$$

—— • ——

$$\vdash \begin{array}{l} a \frown (m \frown \dot{\bot} q) \\ e \frown (m \frown q) \\ a \frown (e \frown \dot{\smile} q) \end{array} \qquad (134$$

If an object belongs to a series starting with a second and stands to a third in the series-forming relation, then the third follows the second in this series.

$$\vdash \begin{array}{l} d \frown (c \frown \dot{\bot} q) \\ d \frown (a \frown q) \\ a \frown (c \frown \dot{\smile} q) \end{array} \qquad (132$$

If an object belongs to a series ending with a second and if a third stands to it in the series-forming relation, then the second follows the third in this series.

$$\begin{array}{l} \vdash\!\!\top\; a\frown(m\frown\acute{\smile}\, q) \\ \quad\llcorner\; a\frown(m\frown\acute{\underline{\ }}\, q) \end{array} \qquad (136$$

—— • ——

$$\begin{array}{l} \vdash\!\!\top\!\!\top\; a\frown(m\frown\acute{\smile}\, q) \\ \quad\;\;\llcorner\; e\frown(m\frown q) \\ \quad\llcorner\; a\frown(e\frown\acute{\smile}\, q) \end{array} \qquad (137$$

—— • ——

$$\begin{array}{l} \vdash\!\!\top\!\!\top\; x\frown(y\frown\acute{\smile}\, q) \\ \quad\;\;\llcorner\; x\frown(d\frown\acute{\smile}\, q) \\ \quad\llcorner\; d\frown(y\frown\acute{\smile}\, q) \end{array} \qquad (322$$

If an object (d) belongs both to a series ending with a second (y) and
| to a series of the same relation starting with a third (x), then the second 246b
also belongs to a series starting with the third.

$$\begin{array}{l} \vdash\!\!\top\!\!\top\; d\frown(y\frown\acute{\smile}\, q) \\ \quad\;\;\llcorner\; d\frown(a\frown q) \\ \quad\llcorner\; a\frown(y\frown\acute{\smile}\, q) \end{array} \qquad (285$$

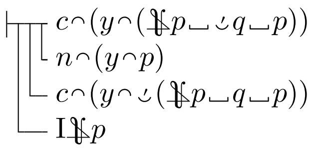
—— • ——

$$\begin{array}{l} \vdash\!\!\top\; a\frown(b\frown\acute{\smile}\, q) \\ \quad\llcorner\; b = a \end{array} \qquad (139$$

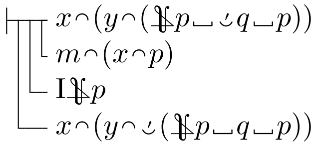
—— • ——

$$\vdash a\frown(a\frown\acute{\smile}\, q) \qquad (140$$

Every object belongs to the series starting with the object itself of any relation.



(335



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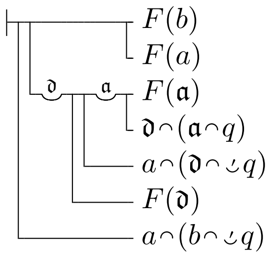

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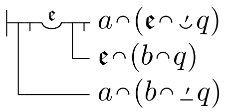
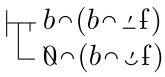


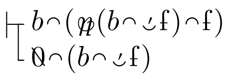
(141

If an object (b) follows a second object (a) in a series, then there is an object which stands to the first (b) in the series-|forming relation and which 247a
belongs to the series of this relation starting with the second (a) (p. 143).



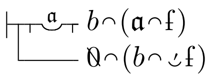
(145

No finite cardinal number follows itself in the cardinal number series (pp. 137 and 144).



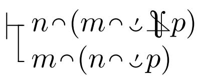
(155

The cardinal number of the members of the cardinal number series ending with a finite cardinal number (b) follows immediately after this cardinal number (b) in the cardinal number series.

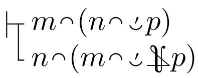


(157

For every finite cardinal number there is a member of the cardinal number series immediately following it.



(303



(304

An object (n) belongs to the series of a (p-)relation starting with a second (m) if the second (m) belongs to the series of the converse relation starting with (n).

$$\vdash \begin{array}{l} a \frown (n \frown \dot{\smile} p) \\ d \frown (a \frown p) \\ \mathrm{I}p \\ d \frown (n \frown \dot{-} p) \end{array} \qquad (242$$

If an object (d) precedes a second (n) in a series whose series-forming
relation is single-valued, and if it stands to a | third (a) in this relation, then 247b
the second (n) belongs to the series of that relation starting with the third
(a).

$$\vdash \begin{array}{l} r \frown (n \frown \dot{\smile} p) \\ n \frown (r \frown \dot{-} p) \\ m \frown (n \frown \dot{\smile} p) \\ \mathrm{I}p \\ m \frown (r \frown \dot{\smile} p) \end{array} \qquad (243$$

If an object (r) which belongs to a series starting with a second (m) whose series-forming relation is single-valued and if a third object (n) belongs to the same series, then the latter (n) belongs to the series of this relation starting with the first (r) or precedes it in the series.

$$\vdash \begin{array}{l} \mathbb{1} \frown (a \frown \dot{\smile} \mathfrak{f}) \\ \mathbb{0} \frown (a \frown \dot{-} \mathfrak{f}) \end{array} \qquad (306$$

If an object follows Zero in the cardinal number series, then it belongs to the cardinal number series starting with One.

$$\vdash \begin{array}{l} \mathbb{0} \frown (a \frown \dot{\smile} \mathfrak{f}) \\ \mathbb{0} \frown (n \frown \dot{\smile} \mathfrak{f}) \\ a \frown (n \frown \dot{\smile} \mathfrak{f}) \end{array} \qquad (307$$

If an object belongs to the cardinal number series ending with a finite cardinal number, then it is itself a finite cardinal number.

$$\vdash \begin{array}{l} y \frown (y \frown \dot{-} q) \\ i \frown (i \frown \dot{-} q) \\ \mathrm{I}q \\ i \frown (y \frown \dot{\smile} q) \end{array} \qquad (296$$

If an object (y) belongs to a series starting with a second (i) whose
series-forming relation is single-valued and if the second object (i) follows
itself in the series of this relation, | then the first (y) also follows itself. 248a

$$\begin{array}{l} \vdash\!\top F(\infty) \\ \quad \llcorner F(\wp(0 \frown 1\!\!\!\!/ \smile \mathfrak{f})) \end{array} \qquad (205$$

—— • ——

$$\vdash \infty \frown (\infty \frown \mathfrak{f}) \qquad (165$$

Endlos immediately follows itself in the cardinal number series.

$$\begin{array}{l} \vdash\!\top\!\top \infty = \wp\dot{\varepsilon}\left(\begin{array}{l}\top\, \varepsilon \frown v \\ \llcorner\!\top\, \varepsilon \frown u\end{array}\right) \\ \quad \llcorner\, \infty = \wp u \\ \quad \llcorner\, 0 \frown (\wp v \frown \smile \mathfrak{f}) \end{array} \qquad (172$$

If Endlos is the cardinal number of a concept and if the cardinal number of another concept is finite, then Endlos is the cardinal number of the concept *falling under the first or under the second concept* (p. 154).

$$\vdash\!\top\, 0 \frown (\infty \frown \smile \mathfrak{f}) \qquad (167$$

Endlos is not a finite cardinal number.

$$\begin{array}{l} \vdash\!\top e \frown (a \frown q) \\ \quad \llcorner\, e \frown (a \frown (u \frown\!\!\!\!- q)) \end{array} \qquad (188$$

—— • ——

$$\begin{array}{l} \vdash\!\top \mathrm{I}(u \frown\!\!\!\!- q) \\ \quad \llcorner\, \mathrm{I}q \end{array} \qquad (189$$

—— • ——

$$\begin{array}{l} \vdash\!\top x \frown (y \frown \underline{\frown} q) \\ \quad \llcorner\, x \frown (y \frown \underline{\frown}(u \frown\!\!\!\!- q)) \end{array} \qquad (194$$

—— • ——

$$\begin{array}{l} \vdash\!\top x \frown (y \frown \smile q) \\ \quad \llcorner\, x \frown (y \frown \smile(u \frown\!\!\!\!- q)) \end{array} \qquad (201$$

—— • ——

$$\begin{array}{l} \vdash\!\top y \frown u \\ \quad \llcorner\, d \frown (y \frown (u \frown\!\!\!\!- q)) \end{array} \qquad (191$$

—— • ——

$$\begin{array}{l} \vdash\!\top\!\top d \frown (a \frown (u \frown\!\!\!\!- q)) \\ \quad \llcorner\, d \frown (a \frown q) \\ \quad \llcorner\, a \frown u \end{array} \qquad (197$$

—— • ——

| 248b

$$\begin{array}{l} \vdash\!\top\!\overset{\mathfrak{q}}{\smile}\!\top\!\top\!\top\!\overset{\mathfrak{a}}{\smile}\!\top\, \dot{\varepsilon}(-\, \varepsilon \frown u) = \mathfrak{a} \frown 1\!\!\!\!/ \smile \mathfrak{q} \\ \qquad \llcorner\overset{\mathfrak{d}}{\smile}\!\top\, \mathfrak{d} \frown u \\ \qquad\quad \llcorner\overset{\mathfrak{e}}{\smile}\!\top\, \mathfrak{d} \frown (\mathfrak{e} \frown \mathfrak{q}) \\ \qquad \llcorner\overset{\mathfrak{i}}{\smile}\!\top\, \mathfrak{i} \frown (\mathfrak{i} \frown \underline{\frown}\, \mathfrak{q}) \\ \qquad \llcorner\, \mathrm{I}\mathfrak{q} \\ \quad \llcorner\, \infty = \wp u \end{array} \qquad (207$$

If Endlos is the cardinal number of objects falling under a concept, then the latter can be ordered into a non-branching series which starts with a specific object and, without looping back into itself, proceeds endlessly (p. 160).

$$\vdash\!\top F(o;a)$$
$$\quad\llcorner F(\dot{\varepsilon}(o\frown(a\frown\varepsilon))) \qquad (249$$

——— • ———

$$\vdash\!\top o;a=e;i$$
$$\quad\llcorner o=e$$
$$\quad\llcorner a=i \qquad (251$$

If an object coincides with a second and a third coincides with a fourth, then the pair consisting of the first and the third coincides with that consisting of the second and the fourth.

$$\vdash o\frown(a\frown q)=q\frown(o;a) \qquad (215$$

——— • ———

$$\vdash\!\top x=d$$
$$\quad\llcorner m;x=c;d \qquad (219$$

If one pair coincides with another, then the second member of the first coincides with the second member of the second.

$$\vdash\!\top m=c$$
$$\quad\llcorner m;x=c;d \qquad (220$$

——— • ———

$$\vdash\!\top f(m,x)=f(c,d)$$
$$\quad\llcorner m;x=c;d \qquad (221$$

——— • ———

249a

$$\vdash\!\top D\frown(A\frown(p\asymp q))$$
$$\quad\llcorner\!\overset{\mathfrak{a}}{\smile}\!\overset{\mathfrak{o}}{\smile}\!\overset{\mathfrak{d}}{\smile}\!\overset{\mathfrak{c}}{\smile}\!\top \mathfrak{c}\frown(\mathfrak{o}\frown p)$$
$$\quad\llcorner D=\mathfrak{c};\mathfrak{d}$$
$$\quad\llcorner \mathfrak{d}\frown(\mathfrak{a}\frown q)$$
$$\quad\llcorner A=\mathfrak{o};\mathfrak{a} \qquad (213$$

——— • ———

249b

$$\vdash\!\top x\frown(a\frown q)$$
$$\quad\llcorner m;x\frown(o;a\frown(p\asymp q)) \qquad (224$$

——— • ———

$$\vdash\!\top m\frown(o\frown p)$$
$$\quad\llcorner m;x\frown(o;a\frown(p\asymp q)) \qquad (225$$

249

$$
\begin{array}{l}
F(n,y) \\
\mathfrak{o}\ \mathfrak{a}\quad F(\mathfrak{o},\mathfrak{a}) \\
x\frown(\mathfrak{a}\frown q) \\
m\frown(\mathfrak{o}\frown p) \\
\mathfrak{c}\ \mathfrak{d}\ \mathfrak{o}\ \mathfrak{a}\quad F(\mathfrak{o},\mathfrak{a}) \\
\mathfrak{d}\frown(\mathfrak{a}\frown q) \\
\mathfrak{c}\frown(\mathfrak{o}\frown p) \\
F(\mathfrak{c},\mathfrak{d}) \\
m;x\frown(n;y\frown\dot{-}(p\asymp q))
\end{array}
\qquad (231
$$

——— • ———

| 249a

$$
\begin{array}{l}
x\frown(d\frown\dot{-}q) \\
m;x\frown(c;d\frown\dot{-}(p\asymp q))
\end{array}
\qquad (233
$$

If a pair follows a second one in the series of a coupled relation, then the second member of the first pair (d) follows the second member of the second pair (x) in a series whose series-forming relation is the second member of the coupled relation. | 249b

$$
\begin{array}{l}
x\frown(d\frown\dot{\smile}q) \\
m;x\frown(c;d\frown\dot{\smile}(p\asymp q))
\end{array}
\qquad (234
$$

——— • ———

$$
\begin{array}{l}
m\frown(b\frown\dot{-}p) \\
m;x\frown(b;d\frown\dot{-}(p\asymp q))
\end{array}
\qquad (244
$$

——— • ———

$$
\begin{array}{l}
m\frown(b\frown\dot{\smile}p) \\
m;x\frown(b;d\frown\dot{\smile}(p\asymp q))
\end{array}
\qquad (246
$$

——— • ———

| 249

$$
\begin{array}{l}
F(n,y) \\
F(m,x) \\
\mathfrak{c}\ \mathfrak{d}\ \mathfrak{o}\ \mathfrak{a}\quad F(\mathfrak{o},\mathfrak{a}) \\
\mathfrak{d}\frown(\mathfrak{a}\frown q) \\
\mathfrak{c}\frown(\mathfrak{o}\frown p) \\
F(\mathfrak{c},\mathfrak{d}) \\
m;x\frown(n;y\frown\dot{\smile}(p\asymp q))
\end{array}
\qquad (257
$$

——— • ———

$$
\begin{array}{l}
\mathrm{I}(p\asymp q) \\
\mathrm{I}p \\
\mathrm{I}q
\end{array}
$$

If a relation is single-valued and so is a second, then the relation formed by coupling the first and the second is also single-valued. | 250a

$$\vdash\; c;d \frown (o; a \frown (p \asymp q))$$
$$d \frown (a \frown q)$$
$$c \frown (o \frown p) \qquad (208$$

If an object (c) stands to a second (o) in a (p-)relation and if a third object (d) stands to a fourth (a) in a second (q-)relation, then the pair consisting of the first and third object ($c; d$) stands to the pair consisting of the second and fourth ($o; a$) in the relation formed by coupling the first and the second relation.

$$\vdash\; A \frown (o; a \frown \smile(p \asymp q))$$
$$d \frown (a \frown q)$$
$$c \frown (o \frown p)$$
$$A \frown (c; d \frown \smile(p \asymp q)) \qquad (209$$

——— • ———

$$\vdash\; x; m \frown (y; n \frown \smile(q \asymp p))$$
$$m; x \frown (n; y \frown \smile(p \asymp q)) \qquad (258$$

——— • ———

$$\vdash\; F(A \frown (b; d \frown \smile t))$$
$$F(b \frown (d \frown (A \nwarrow t))) \qquad (247$$

——— • ———

$$\vdash\; \mathrm{I}(m; x \nwarrow (p \asymp q))$$
$$\mathfrak{i}:\; \mathfrak{i} \frown (\mathfrak{i} \frown \dot- p)$$
$$m \frown (\mathfrak{i} \frown \smile p)$$
$$\mathrm{I}p$$
$$\mathrm{I}q \qquad (253$$

——— • ———

$$\vdash\; F(o \frown (a \frown (A \nwarrow t)))$$
$$F(A \frown (o; a \frown \smile t)) \qquad (210$$

——— • ———

$$\vdash\; m \frown (x \frown (m; x \nwarrow t)) \qquad (238$$

——— • ———

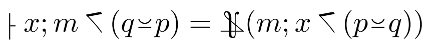
$$\vdash\; x \frown (d \frown \smile q)$$
$$c \frown (d \frown (m; x \nwarrow (p \asymp q))) \qquad (235$$

——— • ———

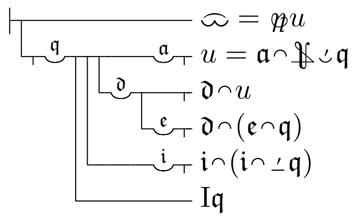
$$\vdash\; o \frown (a \frown (A \nwarrow (p \asymp q)))$$
$$d \frown (a \frown q)$$
$$c \frown (o \frown p)$$
$$c \frown (d \frown (A \nwarrow (p \asymp q))) \qquad (211$$

——— • ———

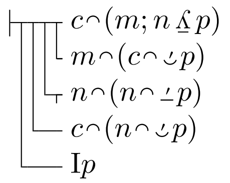
| 250b

 (259

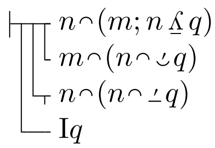
— • —

 (263

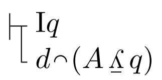
Endlos is the cardinal number of the objects falling under a concept if they can be ordered into a series which starts with a certain object and proceeds endlessly, without branching and without looping back into itself (p. 179).

 (274

An object (c) belongs to the series of a relation running from a second (m) to a third (n) if this relation is single-valued, if the third object (n) does not follow itself in the series of this relation, and if finally the first object (c) belongs to the series of this relation starting with the second (m) as well as the one ending with the third (n).

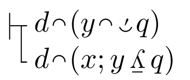
 (344

— • —

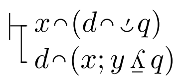
 (365

— • —

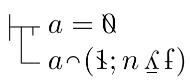
| 251a

 (269

— • —

 (270

If an object belongs to a series running from a second to a third, then it belongs to the series of the same relation starting with the second.

 (312

If an object belongs to the cardinal number series running from One to a second object, then it is distinct from Zero.

$$\vdash\!\!\begin{array}{l} n = \not{p}(\text{1̸}; n \,\underline{\measuredangle}\, \mathfrak{f}) \\ \mathfrak{O} \frown (n \frown \smile \mathfrak{f}) \end{array} \qquad (314$$

Any finite cardinal number is the cardinal number of members of the cardinal number series running from One to itself.

$$\vdash\!\!\begin{array}{l} x \frown (y \frown \smile q) \\ d \frown (x; y \,\underline{\measuredangle}\, q) \end{array} \qquad (323$$

—— • ——

$$\vdash\!\!\begin{array}{l} y \frown (y \frown \perp q) \\ d \frown (x; y \,\underline{\measuredangle}\, q) \end{array} \qquad (271$$

—— • ——

$$\vdash\!\!\begin{array}{l} r = x \\ r \frown (x; x \,\underline{\measuredangle}\, q) \end{array} \qquad (282$$

—— • ——

| 251b

$$\vdash\!\!\begin{array}{l} \not{p}(x; y \,\underline{\measuredangle}\, q) = \not{p}(\text{1̸}; n \,\underline{\measuredangle}\, \mathfrak{f}) \\ y \frown (y \frown \perp q) \\ \mathrm{I}q \\ x; \text{1̸} \frown (y; n \frown \smile (q \,\underline{\smile}\, \mathfrak{f})) \end{array} \qquad (298$$

(p. 202.)

$$\vdash \mathfrak{O} \frown (\not{p}(x; y \,\underline{\measuredangle}\, q) \frown \smile \mathfrak{f}) \qquad (325$$

The cardinal number of members of a series running from an object to an object is finite.

$$\vdash\!\!\begin{array}{l} \mathfrak{O} \frown (\not{p}u \frown \smile \mathfrak{f}) \\ \smile^{\mathfrak{A}}\smile^{\mathfrak{q}}\; u = \mathfrak{A} \,\underline{\measuredangle}\, \mathfrak{q} \end{array} \qquad (327$$

If the objects falling under a concept can be ordered into a series that runs from a specific object to a specific object, then their cardinal number is finite.

$$\vdash\!\!\begin{array}{l} \dot{\varepsilon} g(\varepsilon) = A \,\underline{\measuredangle}\, q \\ \smile^{\mathfrak{a}}\; g(\mathfrak{a}) = (\text{—}\, \mathfrak{a} \frown (A \,\underline{\measuredangle}\, q)) \end{array} \qquad (340$$

—— • ——

$$\vdash\!\!\begin{array}{l} \smile^{\mathfrak{A}}\smile^{\mathfrak{q}}\; \dot{\varepsilon}(\text{—}\, \varepsilon \frown u) = \mathfrak{A} \,\underline{\measuredangle}\, \mathfrak{q} \\ \mathfrak{O} \frown (\not{p}u \frown \smile \mathfrak{f}) \end{array} \qquad (348$$

If the cardinal number of objects falling under a concept is finite, then they can be ordered into series that runs from a specific object to a specific object (p. 224).